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Anomalous transport arising from nonlinear resistive pressure-driven modes in a plasma

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Anomalous transport caused by fluctuations of resistive pressure-driven modes is discussed within the framework of magnetohydrodynamics (MHD). The nonlinear-reduced equations describing fluctuations localized near a particular magnetic field line are derived for tokamak and reversed-field-pinch (RFP) plasmas, taking into account nonzero viscosity and heat conductivity. For an ideally stable but resistively slightly unstable plasma, the anomalous transport is caused particularly by convective motions. The convection is studied as bifurcation from the linearly unstable equilibrium and the expression of the anomalous transport in a tokamak plasma is obtained as a function of the mean pressure gradient near the critical point. In order to evaluate the effects of the convection on the anomalous transport under various conditions, the reduced equations are also solved numerically. It is found that Nusselt number, that is, the ratio of the total heat conductivity including the anomalous heat transport to the classical collisional heat conductivity, is significantly large under some conditions. This partially accounts for the large heat losses in controlled thermonuclear fusion devices.

I. INTRODUCTION

In a magnetically confined plasma, various types of fluctuations affect the global behavior of the plasma through enhancement of the transport of mass, heat, and magnetic fields. The collisions of ions and electrons always generate some diffusion in plasmas, which is called the collisional diffusion. However, the transport caused by the fluctuations, which we call the anomalous transport, has in many cases similar or even greater effects on the plasma than the collisional diffusion. The anomalous transport, therefore, has been a subject of much interest in recent years.

In this paper, we will consider "low-level" or "weak" fluctuations[1] or the fluctuations with small amplitude that vary on faster time scales and smaller spatial scales than the mean fields in a plasma. The fluctuations commonly observed in well-confined plasmas are always weak, typically 1% of magnetic field fluctuations in tokamaks and 10% in reversed field pinches (RFP's). The steady convection or the saturated islands on a rational magnetic surface and plasma turbulence are examples of the weak fluctuations that we are particularly interested in.

One of the most intriguing problems that is believed to be related to the weak fluctuations in a magnetically confined plasma is the spontaneous reversal of the toroidal magnetic field in RFP experiments.[2] Taylor suggested[3] that a slightly resistive plasma minimizes its energy through a magnetic reconnection process, subject to the constraint that the total magnetic helicity of the plasma be conserved. Although he did not specify any particular dynamical process of the plasma relaxation, the predicted minimum energy state agrees well with the experimentally observed equilibrium state of an RFP plasma. Recently, it has been demonstrated[1,4,5] that the dynamical description of a plasma based on the nonlinear resistive magnetohydrodynamic (MHD) equations with the presence of weak fluid fluctuations successfully describes the relaxation of a plasma to a state similar to the one predicted by Taylor.

Another interesting phenomenon associated with the weak fluctuations is the anomalous heat transport.[6] For instance, turbulence in an RFP and steady convection on some rational magnetic surfaces in a tokamak enhance the heat conduction across the magnetic flux surfaces and deteriorate energy confinement of the plasmas. Thus it is important to know how the anomalous heat transport is related to a given set of macroscopic profiles of the plasma, such as the pressure profile, in order to determine the energy confinement properties of the plasma. Determination of the dependence of the anomalous heat transport on such macroscopic conditions is one of the main goals of this paper.

The class of the weak fluctuations with which we are particularly concerned here is the one generated by the resistive g mode or, more precisely, the resistive fast interchange mode,[7] which is the instability caused by the pressure gradient acting against the curvature of the magnetic field lines in a plasma with finite resistivity. These resistive pressure-driven modes are expected to be present in tokamaks and RFP's. Recently, Hameiri[6] discussed the turbulent heat conduction without specifying any particular modes and derived some of its general properties. In this paper we will consider the anomalous heat transport specifically caused by the resistive g-mode fluctuations and we will determine a more precise characterization of the transport.

For the resistive g mode, the free energy source is the mean pressure gradient and the energy sink is the collisional diffusion. Therefore, when the mean pressure gradient exceeds a critical value (which we call the linear stability limit and denote by D_L in terms of the parameter D defined in Sec. III; the precise definition of D_L is given in Appendix B), modes localized on a rational surface begin to grow. If the mean pressure gradient is, however, not too large compared to the critical value, and interactions of modes localized on different rational surfaces are ignorable, then these modes eventually saturate with finite amplitude, rather than leading to fully developed turbulence. These saturated modes or steady convection cells within a boundary layer are analogous to the Bénard convection cells of fluid dynamics and enhance the heat transport across the magnetic surface. In this paper, we will estimate this anomalous heat transport, using nonlinear bifurcation analysis.[8] In this method, the set of the nonlinear equations of the fluctuations is expanded in terms of a small amplitude and the anomalous heat transport is calculated to its lowest order. Also presented is the complete algorithm to determine all the higher-order corrections to the calculated anomalous heat transport. We will also show the numerical results that determine the dependence of the anomalous heat transport on the mean quantities under various conditions.

This paper consists of seven sections. In Sec. II the set of equations governing the mean quantities obtained by Hameiri[6] is reviewed and extended. In Sec. III we derive the nonlinear reduced equations describing the fluctuations due to the resistive g modes with finite viscosity and heat conductivity for both an RFP and a tokamak. In Sec. IV the relationship between the anomalous electric field, which is partly related to the dynamo effect in an RFP, and the anomalous heat transport is discussed. In Sec. V the dependence of the anomalous transport due to the convection on the mean pressure gradient is derived analytically near the critical pressure gradient for a tokamak plasma, and the numerical simulations are presented in Sec. VI. Finally, Sec. VII contains our conclusions, where the scaling law of the anomalous heat conductivity is presented.

II. EQUATIONS OF THE MEAN MOTION

We begin by reviewing the equations describing evolution of the mean quantities, slightly generalizing the earlier work[6] by taking into account the parallel viscosity.[9] We start from the following resistive MHD equations:

$$\rho\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\right)\mathbf{v} = -\nabla p + \mathbf{J}\times\mathbf{B} - \nabla\cdot\mathbf{\Pi}, \tag{1a}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla\times(\eta\mathbf{J} - \mathbf{v}\times\mathbf{B}) = 0, \tag{1b}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0, \tag{1c}$$

$$\frac{\partial p}{\partial t} + \mathbf{v}\cdot\nabla p + \gamma p\nabla\cdot\mathbf{v} = (\gamma-1)[\eta\,\mathbf{J}\cdot\mathbf{J} + \nabla\cdot(\boldsymbol{\kappa}\nabla T) - \mathbf{\Pi}:\nabla\mathbf{v}], \tag{1d}$$

where

$$p = \rho T, \tag{1e}$$

$$\mathbf{\Pi} = -3\mu_\parallel(\hat{\mathbf{b}}\hat{\mathbf{b}} - \tfrac{1}{3}\mathsf{I})\lambda - \mu_\perp\boldsymbol{\sigma}, \tag{1f}$$

$$\lambda = \hat{\mathbf{b}}\cdot[(\hat{\mathbf{b}}\cdot\nabla)\mathbf{v}] - \tfrac{1}{3}\nabla\cdot\mathbf{v}. \tag{1g}$$

In Eqs. (1), symbols have their usual meanings: ρ, p, T, and $\mathbf{v}$ are the mass density, pressure, temperature, and velocity, respectively. The magnetic field $\mathbf{B}$ satisfies $\nabla\cdot\mathbf{B}=0$, $\mathbf{J}=\nabla\times\mathbf{B}$ is the current density, γ is the ratio of the specific heats, η is the resistivity, and $\boldsymbol{\kappa}$ is the heat conductivity tensor. Here, we only consider the perpendicular component $\kappa_\perp$ and the parallel component $\kappa_\parallel$ of the conductivity tensor $\boldsymbol{\kappa}$ so that $\boldsymbol{\kappa}\nabla = \kappa_\parallel\nabla_\parallel + \kappa_\perp\nabla_\perp$, where $\nabla_\parallel$ and $\nabla_\perp$ denote derivatives parallel and perpendicular to the magnetic field $\mathbf{B}$, respectively. Equation (1f) is the definition of the stress tensor $\mathbf{\Pi}$, where I denotes the unit tensor, $\hat{\mathbf{b}} = \mathbf{B}/|\mathbf{B}|$, and $\boldsymbol{\sigma}$ is the rate-of-strain tensor defined by

$$\sigma_{ij} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\nabla\cdot\mathbf{v}$$

for the (i,j) component in Cartesian coordinates. Here, δ_{ij} indicates Kronecker's delta, that is, $\delta_{ij}=1$ if $i=j$ and $\delta_{ij}=0$ if $i\neq j$. Also $\nabla\cdot\mathbf{\Pi} = \Sigma_i\partial\Pi_{ij}/\partial x_i$ is the divergence of the tensor $\mathbf{\Pi}$ and $\mathbf{\Pi}:\nabla\mathbf{v}$ is the contraction defined as $\mathbf{\Pi}:\nabla\mathbf{v} = \Sigma_i\Sigma_j\Pi_{ij}\partial v_i/\partial x_j$. It follows that

$$\mathbf{\Pi}:\nabla\mathbf{v} = -3\mu_\parallel\lambda^2 - (\mu_\perp/2)\mathrm{tr}\,\boldsymbol{\sigma}^2,$$

where tr denotes the trace of the tensor. We assume that the plasma is confined in either a toroidal or a cylindrical vessel with minor radius a. For the boundary conditions, the perfectly conducting wall is assumed.

We now consider fluctuating solutions of Eqs. (1). Denoting by $\langle\ \rangle$ either an ensemble average or an average over the small space and fast time scales of the fluctuations, one can write every physical quantity as a sum of the mean part, denoted by subscript 0, and the fluctuating part, denoted by subscript 1. For example, the magnetic field $\mathbf{B}=\mathbf{B}_0+\mathbf{B}_1$, where $\mathbf{B}_0 = \langle\mathbf{B}\rangle$. Since the following discussion in this section does not depend on the particular choice of the averaging operation $\langle\ \rangle$, the actual form of the averaging is not specified here. Using the ordering assumptions specified in Ref. 5 for low-level fluid fluctuations, together with the assumptions that $\mu_\parallel = O(1)$ and $\mu_\perp = O(\eta)$, we obtain the following set of equations of the mean quantities:

$$\nabla p_0 = \mathbf{J}_0\times\mathbf{B}_0, \tag{2a}$$

$$\frac{\partial\mathbf{B}_0}{\partial t} + \nabla\times(\eta\mathbf{J}_0 - \boldsymbol{\epsilon} - \mathbf{u}_0\times\mathbf{B}_0) = 0, \tag{2b}$$

$$\frac{\partial\rho_0}{\partial t} + \nabla\cdot(\rho_0\mathbf{u}_0) = 0, \tag{2c}$$

$$\begin{aligned}\frac{\partial p_0}{\partial t} &+ \mathbf{u}_0\cdot\nabla p_0 + \gamma p_0(\nabla\cdot\mathbf{u}_0),\\ &= (\gamma-1)[\eta\mathbf{J}_0\cdot\mathbf{J}_0 - \boldsymbol{\epsilon}\cdot\mathbf{J}_0 + \nabla\cdot(\boldsymbol{\kappa}\nabla T_0)\\ &\quad - \nabla\cdot(p_0\langle s_1\mathbf{v}_1\rangle) + f],\end{aligned} \tag{2d}$$

where

$$f = -\nabla\cdot\langle(p_1+\mathbf{B}_0\mathbf{B}_1)\mathbf{v}_1 - (\mathbf{v}_1\cdot\mathbf{B}_1)\mathbf{B}_0 + \tfrac{1}{2}\rho_0\mathbf{v}^2\cdot\mathbf{v}\rangle,$$

$$\boldsymbol{\epsilon} = \langle\mathbf{v}_1\times\mathbf{B}_1\rangle - (1/\rho_0)\langle\rho_1\mathbf{v}_1\rangle\times\mathbf{B}_0,$$

$$\mathbf{u}_0 = \mathbf{v}_0 + (1/\rho_0)\langle\rho_1\mathbf{v}_1\rangle$$

and the specific entropy $s\equiv[1/(\gamma-1)]\log(p/\rho^\gamma)$. In this derivation, we used the following relation[6]:

$$\begin{aligned}\boldsymbol{\epsilon}\cdot\mathbf{J}_0 &+ \rho_0\langle s_1\mathbf{v}_1\rangle\cdot\nabla T_0\\ &= -[\langle\eta\mathbf{J}_1\cdot\mathbf{J}_1\rangle + (1/T_0)\langle\boldsymbol{\kappa}\nabla T_1\cdot\nabla T_1\rangle + 3\mu_\parallel\langle\lambda_1^2\rangle\\ &\quad + (\mu_\perp/2)\mathrm{tr}\langle\boldsymbol{\sigma}_1^2\rangle] + f.\end{aligned} \tag{3}$$

We will derive this equation (with $f=0$), particularly for the resistive g-mode fluctuations, in Sec. IV, although Eq. (3) is derived under more general conditions in Ref. 6. From

Eqs. (2), it is clear that ϵ and $p_0\langle s_1 \mathbf{v}_1\rangle$ are the electric field and the heat flux caused by the fluctuations, respectively. We, therefore, call ϵ the anomalous electric field and $p_0\langle s_1 \mathbf{v}_1\rangle$ the anomalous heat flux.

Equations (2a)–(2d) are the closed set of equations used to evolve the mean quantities if the anomalous electric field ϵ, the anomalous heat flux $p_0\langle s_1 \mathbf{v}_1\rangle$, and the function f are given. We note that f vanishes for localized fluctuations, as will be discussed. We also note that the quantity $\mathbf{u}_0$ is determined, as in the Grad–Hogan model[10] for diffusion, by the requirement that all mean quantities evolve through a sequence of magnetostatic equilibrium states satisfying Eq. (2a). The dependence of the anomalous electric field ϵ on the mean profiles was determined by Bhattacharjee and Hameiri[1,5] for the case of tearing-mode-induced turbulence. One of the goals of this paper is to estimate the anomalous heat flux $p_0\langle s_1 \mathbf{v}_1\rangle$ from the equations of the resistive g-mode fluctuations.

III. REDUCED EQUATIONS OF THE FLUCTUATIONS

In this section we will derive the equations of fluctuating quantities arising from the resistive g mode with a finite heat conductivity. This is an extension of the reduced equations of Bhattacharjee and Hameiri.[1,5] Assuming that the modes are localized along a mean magnetic field line, we apply the scaling of the resistive fast interchange mode in the linear stability theory[7] to the nonlinear system: If the following operators are applied to the fluctuating quantities, then $\nabla_\perp = O(1/\delta)$, $\mathbf{B}_0\cdot\nabla = O(1)$, and $\partial/\partial t = O(1)$, where the perpendicular derivatives $\nabla_\perp \equiv \nabla - \mathbf{b}(\mathbf{B}_0\cdot\nabla)$ and $\mathbf{b} = \mathbf{B}_0/|\mathbf{B}_0|^2$. Here, δ is a small parameter measuring the localizatin of the mode, taken to be $O(\sqrt{\eta})$. The fluctuating quantities $\mathbf{v}_1$, $\mathbf{B}_1$, p_1, and T_1 are assumed to be $O(\delta)$, whereas all the mean quantities and their spatial derivatives are taken to be $O(1)$, except for $\mathbf{v}_0 = O(\delta^2)$, and their time derivatives are taken to be $O(\delta^2)$. Here, for simplicity, we take the parallel viscosity coefficient $\mu_\parallel$ to be smaller than $O(1)$ so that it does not enter the equation of the fluctuations. The more complete version of the mode equations, including the parallel viscosity coefficient $\mu_\parallel$, is derived in Appendix A. For other diffusion coefficients, we take $\kappa_\perp, \mu_\perp$ to be $O(\delta^2)$ and $\kappa_\parallel$ to be $O(1)$. Following Ref. 5, we obtain the set of equations for the fluctuations:

$$\frac{\partial A}{\partial t} + (\nabla_\perp A \times \nabla_\perp \phi)\mathbf{b} = \eta\Delta_\perp A + \mathbf{B}_0\cdot\nabla\phi, \tag{4a}$$

$$\begin{aligned}\rho_0 \frac{d}{dt}\Delta_\perp \phi = {} & (\mathbf{B}_0 + \mathbf{B}_{1\perp})\cdot\nabla(\Delta_\perp A) \\ & - 2\mathbf{b}\times\nabla(p_0 + \tfrac{1}{2}\mathbf{B}_0^2)\cdot\nabla p_1 \\ & - 2\mathbf{b}\cdot\nabla(\tfrac{1}{2}\mathbf{B}_0^2)(\Delta_\perp A) + \mu_\perp \Delta_\perp^2 \phi,\end{aligned} \tag{4b}$$

$$\begin{aligned}\frac{dp_1}{dt} + \mathbf{v}_{1\perp}\cdot\nabla p_0 = {} & \frac{\gamma p_0}{\gamma p_0 + \mathbf{B}_0^2}[2\mathbf{v}_{1\perp}\cdot\nabla(p_0 + \tfrac{1}{2}\mathbf{B}_0^2) \\ & + (\mathbf{B}_0 + \mathbf{B}_{1\perp})\cdot\nabla v_1 + \eta\Delta p_1] \\ & + \frac{(\gamma-1)\mathbf{B}_0^2}{\gamma p_0 + \mathbf{B}_0^2}\nabla\cdot[(\kappa_\parallel \nabla_\parallel + \kappa_\perp \nabla_\perp)T_1],\end{aligned} \tag{4c}$$

$$\begin{aligned}\frac{d}{dt}T_1 + \mathbf{v}_{1\perp}\cdot\nabla T_0 = {} & \frac{(\gamma-1)T_0}{\mathbf{B}_0^2 + \gamma p_0}[2\mathbf{v}_{1\perp}\cdot\nabla(p_0 + \tfrac{1}{2}\mathbf{B}_0^2) \\ & + (\mathbf{B}_0 + \mathbf{B}_{1\perp})\cdot\nabla v_1 + \eta\Delta p_1] \\ & + \frac{(\gamma-1)}{\rho_0}\frac{\mathbf{B}_0^2 + p_0}{\gamma p_0 + \mathbf{B}_0^2}\nabla \\ & \cdot[(\kappa_\parallel \nabla_\parallel + \kappa_\perp \nabla_\perp)T_1],\end{aligned} \tag{4d}$$

$$\rho_0 \frac{d}{dt} v = (\mathbf{B}_0 + \mathbf{B}_{1\perp})\cdot\nabla p_1 + \mathbf{B}_1\cdot\nabla p_0 + \mu_\perp \Delta_\perp v, \tag{4e}$$

where A, ϕ, and v are defined by

$$\mathbf{B}_1 = \nabla_\perp A \times \mathbf{b} - p_1 \mathbf{b},$$

$$\mathbf{v}_1 = \nabla_\perp \phi \times \mathbf{b} - v\mathbf{b},$$

the subscript $\perp$ denotes the component perpendicular to the mean magnetic field $\mathbf{B}_0$, and $\Delta_\perp = \nabla_\perp^2$. In the derivation of Eqs. (4), the relation $\rho_1/\rho_0 = p_1/p_0 - T_1/T_0$ is used. Either Eq. (4c) or Eq. (4d) may be replaced by the following entropy equation:

$$\frac{ds_1}{dt} + (\mathbf{v}_{1\perp}\cdot\nabla)s_0 = \frac{1}{p_0}\nabla\cdot[(\kappa_\parallel \nabla_\parallel + \kappa_\perp \nabla_\perp)T_1], \tag{4f}$$

where $s_1 = \gamma T_1/(\gamma-1)T_0 - p_1/p_0$.

We will further simplify these equations by assuming that the plasma is confined in a cylinder and all the mean quantities depend only on its radius r. It is convenient in this case to use the following independent variables:

$$\begin{aligned} & x = |\sigma|^{1/2}[(r - r_0)/r_0], \\ & y = (B_\theta/r_0 B)|\sigma|^{1/2}[z - \mu(r)\theta], \\ & \tilde{\theta} = |\sigma|\theta, \quad \tau = (B_\theta|\sigma|/r_0\sqrt{\rho_0})t. \end{aligned} \tag{5}$$

Here, (r,θ,z) denote the usual polar coordinates and r_0 denotes a particular radius in the vicinity of which we consider the motion of the modes. The azimuthal and longitudinal components of the mean magnetic field $\mathbf{B}_0$ are denoted by B_θ and B_z, respectively. We note that the $\tilde{\theta}$ direction (with fixed x and y) is the direction along the equilibrium magnetic field line $\mathbf{B}_0(r)$ and not the azimuthal direction anymore. The following definitions are also used in Eqs. (5): $B = |\mathbf{B}_0|$, $\mu = rB_z/B_\theta$, and $\sigma = B_\theta \mu'/B$, where the prime denotes d/dr. In Eqs. (5), the mean quantities B, B_θ, ρ_0, and σ are evaluated at $r = r_0$. We also introduce the following parameters:

$$\begin{aligned} & D = -2rp_0'/B^2\sigma^2, \quad \beta = 2p_0/B^2, \\ & S = 4B_\theta^2/B^2\sigma^2, \quad R = rB_\theta/\eta\sqrt{\rho_0}, \\ & \Theta = \frac{\gamma}{\gamma-1}\frac{-2r\rho_0}{B^2\sigma^2}T_0', \quad M = \frac{\mu_\perp}{rB_\theta\sqrt{\rho_0}}, \\ & K_\parallel = \frac{\gamma-1}{\gamma}\frac{\kappa_\parallel|\sigma|B_\theta}{rB^2\sqrt{\rho_0}}, \quad K_\perp = \frac{\gamma-1}{\gamma}\frac{\kappa_\perp}{rB_\theta\sqrt{\rho_0}}, \end{aligned} \tag{6}$$

where every quantity is evaluated at $r = r_0$. The linear stability criterion for the ideal modes is given by Suydam's criterion,[11] $D < \frac{1}{4}$. We assume that this stability condition is always satisfied so that the plasma is ideally stable and the instability arises from the finite diffusivity. The dependent variables are defined in the dimensionless form as

$$\tilde{A} = (1/r_0 B_\theta B)A, \quad \tilde{\phi} = (\sqrt{\rho_0}/r_0 B_\theta B)\phi,$$
$$\tilde{p} = (2/B^2|\sigma|^{3/2})p_1, \quad \tilde{v} = (2\sqrt{\rho_0}/B^2|\sigma|^{3/2})v, \tag{7}$$
$$\tilde{T} = \frac{\gamma}{\gamma-1}\frac{2\rho_0}{B^2|\sigma|^{3/2}}T_1, \quad \tilde{s} = \frac{2\rho_0}{B^2|\sigma|^{3/2}}s_1,$$

where $\tilde{s} = \tilde{T} - \tilde{p}$. We note that $\tilde{A}$ and $\tilde{\phi}$ are of order δ^2 and $\tilde{p}$, $\tilde{v}$, $\tilde{T}$, and $\tilde{s}$ are of order δ.

In terms of these new definitions, Eqs. (4) become

$$\frac{d\tilde{A}}{d\tau} = \frac{\partial\tilde{\phi}}{\partial\tilde{\theta}} + \frac{1}{R}\tilde{\Delta}_\perp\tilde{A}, \tag{8a}$$

$$\frac{d}{d\tau}\tilde{\Delta}_\perp\tilde{\phi} = \frac{\partial}{\partial\tilde{\theta}}\tilde{\Delta}_\perp\tilde{A} + \{\tilde{A},\tilde{\Delta}_\perp\tilde{A}\} - \frac{\partial\tilde{p}}{\partial y} + M\tilde{\Delta}_\perp^2\tilde{\phi}, \tag{8b}$$

$$\frac{d}{d\tau}\tilde{p} = \frac{\gamma\beta}{2+\gamma\beta}\left[S\frac{\partial\tilde{\phi}}{\partial y} + \frac{\partial\tilde{v}}{\partial\tilde{\theta}} + \{\tilde{A},\tilde{v}\} + \frac{1}{R}\tilde{\Delta}_\perp\tilde{p}\right] - D\frac{\partial\tilde{\phi}}{\partial y} + \frac{2(\gamma-1)}{2+\gamma\beta}(K_\parallel\tilde{\Delta}_\parallel + K_\perp\tilde{\Delta}_\perp)\tilde{T}, \tag{8c}$$

$$\frac{d}{d\tau}\tilde{T} = \frac{\gamma\beta}{2+\gamma\beta}\left[S\frac{\partial\tilde{\phi}}{\partial y} + \frac{\partial\tilde{v}}{\partial\tilde{\theta}} + \{\tilde{A},\tilde{v}\} + \frac{1}{R}\tilde{\Delta}_\perp\tilde{p}\right] - \Theta\frac{\partial\tilde{\phi}}{\partial y} + \gamma\frac{2+\beta}{2+\gamma\beta}(K_\parallel\tilde{\Delta}_\parallel + K_\perp\tilde{\Delta}_\perp)\tilde{T}, \tag{8d}$$

$$\frac{d\tilde{v}}{d\tau} = \frac{\partial\tilde{p}}{\partial\tilde{\theta}} + \{\tilde{A},\tilde{p}\} + D\frac{\partial\tilde{A}}{\partial y} + M\tilde{\Delta}_\perp\tilde{v}. \tag{8e}$$

Either Eqs. (8c) or (8d) may be replaced by the entropy equation

$$\frac{d}{d\tau}\tilde{s} = (D-\Theta)\frac{\partial\tilde{\phi}}{\partial y} + (K_\parallel\tilde{\Delta}_\parallel + K_\perp\tilde{\Delta}_\perp)\tilde{T}. \tag{8f}$$

Here, the following definitions have been used:

$$\{f,g\} = (\nabla_\perp g\times\nabla_\perp f)\cdot\hat{b} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x},$$
$$\tilde{\nabla}_\perp = \nabla r\left(\frac{\partial}{\partial x} - \tilde{\theta}\frac{\partial}{\partial y}\right) + \frac{\nabla r\times\mathbf{B}_0}{B}\frac{\partial}{\partial y},$$
$$\frac{df}{d\tau} = \frac{\partial f}{\partial\tau} + \{\phi,f\},$$
$$\tilde{\Delta}_\parallel f = \frac{\partial^2 f}{\partial\tilde{\theta}^2} + \left(\left\{\tilde{A},\frac{\partial f}{\partial\tilde{\theta}}\right\} + \frac{\partial}{\partial\tilde{\theta}}\{\tilde{A},f\}\right) + \{\tilde{A},\{\tilde{A},f\}\},$$
$$\tilde{\Delta}_\perp = \left(\frac{\partial}{\partial x} - \frac{\sigma}{|\sigma|}\tilde{\theta}\frac{\partial}{\partial y}\right)^2 + \frac{\partial^2}{\partial y^2}.$$

If we apply the following scale transformations to Eqs. (8), the coefficients $1/R$, M, and $K_\perp$ will be formally replaced by 1, M_R, and K_R, respectively,

$$x\to R^{-1/2}x, \quad y\to R^{-1/2}y, \quad \tilde{A}\to R^{-1}\tilde{A}, \quad \tilde{\phi}\to R^{-1}\tilde{\phi},$$
$$\tilde{p}\to R^{-1/2}\tilde{p}, \quad \tilde{T}\to R^{-1/2}\tilde{T}, \quad \tilde{v}\to R^{-1/2}\tilde{v}, \tag{9}$$
$$\tilde{s}\to R^{-1/2}\tilde{s}.$$

Here, the normalized viscosity M_R and the normalized heat conductivity K_R are defined as $M_R = MR = \mu_\perp/\eta\rho_0$ and $K_R = \gamma K_\perp R = (\gamma-1)\kappa_\perp/\eta\rho_0$. Therefore, the transformed system depends only on M_R and K_R as well as D, Θ, S, β, and $K_\parallel$, but not on the magnetic Reynolds number R.

The set of equations (8) is suitable to describe the resistive g-mode fluctuations in a screw pinch or an RFP, in which the azimuthal (or poloidal) field B_θ and the longitudinal (or toroidal) field B_z are of the same order of magnitude, and the plasma beta β, which is defined as the ratio of the thermal pressure p_0 to the magnetic pressure $B_0^2/2$, is of order 1. However, if we consider a straight cylindrical plasma with low beta and a strong longitudinal field B_z, we may further simplify the set of equations (8). This further simplified system is a reasonable model of a tokamak plasma with a large aspect ratio.

Let ϵ_a be the inverse aspect ratio, i.e., $\epsilon_a = a/L$, where a and L denote the radius and the length of the cylinder, respectively. We assume that ϵ_a is a small number, $\beta\sim B_\theta/B_z\sim\epsilon_a$, and $\sigma\sim\sqrt{\epsilon_a}$, namely, low beta, strong longitudinal field, and low shear. The diffusion coefficients, such as the resistivity η, the perpendicular viscosity ν, and the perpendicular heat conductivity $\kappa_\perp$, are also taken to be of order $\epsilon_a\cdot\delta^2$. Under these assumptions, it follows that $D\sim\Theta\sim R\sim M\sim K_\perp$ are of order 1 and S is of order ϵ_a. It also follows that the safety factor $q(r)$, defined by $q(r)\equiv rB_z(r)/LB_\theta(r)$, is of order 1, which is the case in tokamaks.

In order to focus on the effect of the temperature gradient, we also assume that the density gradient is small, i.e., $r_0\rho_0'/\rho_0 = O(\epsilon_a)$. This assumption makes Eqs. (8c) and (8d) identical to each other to the lowest order, and we have $D = (\gamma-1)\Theta/\gamma$, $\tilde{T} = \gamma\tilde{p}/(\gamma-1)$, and $\tilde{s} = \tilde{p}/(\gamma-1)$. Thus the subsidiary expansion of Eqs. (8) in ϵ_a leads to the following set of equations:

$$\frac{d\tilde{A}}{d\tau} = \frac{\partial\tilde{\phi}}{\partial\tilde{\theta}} + \frac{1}{R}\tilde{\Delta}_\perp\tilde{A}, \tag{10a}$$

$$\frac{d}{d\tau}\tilde{\Delta}_\perp\tilde{\phi} = \frac{\partial}{\partial\tilde{\theta}}\tilde{\Delta}_\perp\tilde{A} + \{\tilde{A},\tilde{\Delta}_\perp\tilde{A}\} - \frac{\partial\tilde{p}}{\partial y} + M\tilde{\Delta}_\perp^2\tilde{\phi}, \tag{10b}$$

$$\frac{d}{d\tau}\tilde{p} = -D\frac{\partial\tilde{\phi}}{\partial y} + \chi\tilde{\Delta}_\perp\tilde{p}. \tag{10c}$$

Here, $\chi = \gamma K_\perp$ and Eq. (8e) of $\tilde{v}$ is decoupled from the set of Eqs. (10). Although we ignored the parallel heat conductivity in the system (10), we can retrieve it simply by replacing $\chi\tilde{\Delta}_\perp$ with $\chi_\parallel\tilde{\Delta}_\parallel + \chi\tilde{\Delta}_\perp$ in Eq. (10c), where $\chi_\parallel = \gamma K_\parallel$. We note that, as in the case of the system (8), the scale transformations (9) replace the coefficients $1/R$, M, and χ in Eqs. (10) with 1, M_R, and K_R, respectively, so that the solutions of the transformed system depend only on the parameters D, M_R, and K_R.

These two reduced sets of equations, i.e., Eqs. (8) and (10), form the basis of our models of the nonlinear resistive pressure driven modes. From the next section on, we will investigate these reduced equations in order to characterize the anomalous transport caused by these fluctuations.

IV. COUPLING OF THE ANOMALOUS ELECTRIC FIELD WITH THE HEAT TRANSPORT

In this section, we will derive Eq. (3) directly from the system (14), which gives a relationship between the anomalous electric field ϵ and the anomalous heat transport

$\langle s_1 v_{1r}\rangle$. First, we transform[12] the independent variables x, y, and $\widetilde{\theta}$ to the new variables $\tilde{x} = x, \tilde{y} = y + \widetilde{\theta}x$, and $\tilde{z} = \widetilde{\theta}$. This transformation changes the derivatives in Eqs. (8) as follows: $\partial/\partial x = \partial/\partial\tilde{x} + \tilde{z}\partial/\partial\tilde{y}$, $\partial/\partial y = \partial/\partial\tilde{y}$, $\partial/\partial\widetilde{\theta} = \partial/\partial\tilde{z} + \tilde{x}\partial/\partial\tilde{y}$, and $\widetilde{\Delta}_\perp = \partial^2/\partial^2\tilde{x} + \partial^2/\partial^2\tilde{y}$. Then, it is readily seen that the well-known linear solution of the resistive fast interchange mode[7] is the $\tilde{z}$-independent solution of the linearized version of the system of equations (8). With the new coordinate system $(\tilde{x},\tilde{y},\tilde{z})$, the boundary conditions of Eqs. (8) are such that all the fluctuating quantities are periodic in $\tilde{y}$ and $\tilde{z}$ with periods $2\delta_y$ and L_z, respectively, and decay rapidly as $|\tilde{x}| \to \infty$. Here, δ_y and L_z are taken to be quantities of $O(1/\sqrt{R})$ and $O(1)$, respectively. (A more detailed discussion on the boundary condition is found in Appendix B.)

For such localized fluctuations, it is natural to define the averaging operation as an average over the small space and fast time scales of the fluctuations[1]: for a fluctuating quantity $f(\tilde{x},\tilde{y},\tilde{z},\tau; r_0,t)$ satisfying the boundary conditions mentioned above, we define

$$\langle f\rangle = \frac{1}{2\Delta\delta_y L_z}\int_{-\infty}^{\infty} d\tilde{x}\int_{-\delta_y}^{\delta_y} d\tilde{y}\int_0^{L_z} d\tilde{z}\, f(\tilde{x},\tilde{y},\tilde{z},\tau;r_0,t),$$

$$\bar{f} = \lim_{T\to\infty}\frac{1}{T}\int_0^T d\tau\, f(\tilde{x},\tilde{y},\tilde{z},\tau;r_0,t),$$

where Δ is the typical width of the mode f in the x direction.

We also define the averaging $\overline{\langle\ \rangle}$ as the combination of these two averagings, i.e., $\overline{\langle f\rangle} \equiv \overline{\langle f\rangle}$. Although the boundary conditions and the averaging operation are defined based on the new coordinate system $(\tilde{x},\tilde{y},\tilde{z})$, we will use both the coordinate system $(x,y,\widetilde{\theta})$ and $(\tilde{x},\tilde{y},\widetilde{\theta})$, using whichever system makes the equations simpler.

We now derive Eq. (3) directly from the reduced equations (8). Adding up $\widetilde{\Delta}_\perp\widetilde{A}\times$Eq. (8a) and $\tilde{\phi}\times$Eq. (8b) and taking the $\langle\ \rangle$ average of the resulting equation yields

$$\begin{aligned}\frac{\partial}{\partial\tau}\frac{1}{2}&(\langle|\widetilde{\nabla}_\perp\widetilde{A}|^2 + |\widetilde{\nabla}_\perp\tilde{\phi}|^2\rangle)\\ &= -\left\langle\tilde{p}\frac{\partial\tilde{\phi}}{\partial y}\right\rangle = \frac{1}{R}\langle|\widetilde{\Delta}_\perp\widetilde{A}|^2\rangle\\ &\quad - M\langle|\widetilde{\Delta}_\perp\tilde{\phi}|^2\rangle.\end{aligned} \tag{11}$$

For the solutions such as saturated modes or stationary turbulence, we can drop the left-hand sides of this equation by taking the long time average in τ:

$$\overline{\left\langle\tilde{p}\frac{\partial\tilde{\phi}}{\partial y}\right\rangle} = -\frac{1}{R}\overline{\langle|\widetilde{\Delta}_\perp\widetilde{A}|^2\rangle} - M\overline{\langle|\widetilde{\Delta}_\perp\tilde{\phi}|^2\rangle}. \tag{12}$$

Similarly, taking the $\overline{\langle\ \rangle}$ average of the combination of equations $\tilde{p}\cdot$[Eq. (8c)] $+ \gamma\beta/(2+\gamma\beta)\cdot$[Eq. (8e)] $+ 2(\gamma-1)/(2+\gamma\beta)\cdot$[Eq. (8f)], we obtain

$$\begin{aligned}&-D\left(\overline{\left\langle\tilde{p}\frac{\partial\tilde{\phi}}{\partial y}\right\rangle} - \frac{2(\gamma-1)}{2+\gamma\beta}\overline{\left\langle\tilde{s}\frac{\partial\tilde{\phi}}{\partial y}\right\rangle} - \frac{\gamma\beta}{2+\gamma\beta}\overline{\left\langle\tilde{v}\frac{\partial\widetilde{A}}{\partial y}\right\rangle}\right) - \frac{2(\gamma-1)}{2+\gamma\beta}\Theta\overline{\left\langle\tilde{s}\frac{\partial\tilde{\phi}}{\partial y}\right\rangle}\\ &= \frac{\gamma\beta}{2+\gamma\beta}S\left(\frac{1}{R}\overline{\langle|\widetilde{\Delta}_\perp\widetilde{A}|^2\rangle} + M\overline{\langle|\widetilde{\Delta}_\perp\tilde{\phi}_\perp|^2\rangle}\right) + \frac{\gamma\beta}{2+\gamma\beta}\frac{1}{R}\overline{\langle|\widetilde{\Delta}_\perp\tilde{p}|^2\rangle}\\ &\quad + \frac{\gamma\beta}{2+\gamma\beta}\overline{\langle|\widetilde{\Delta}_\perp\tilde{v}|^2\rangle} + \frac{2(\gamma-1)}{2+\gamma\beta}\left(K_\parallel\overline{\left\langle\left|\frac{\partial\widetilde{T}}{\partial\widetilde{\theta}} - \{\widetilde{A},\widetilde{T}\}\right|^2\right\rangle} + K_\perp\overline{\langle|\widetilde{\nabla}_\perp\widetilde{T}|^2\rangle}\right).\end{aligned} \tag{13}$$

It is easy to check that the sum of the first three terms of the left-hand side is proportional to $\boldsymbol{\epsilon}\cdot\mathbf{J}_0$ and the fourth term is proportional to $\langle s_1 v_{1r}\rangle$, where $v_{1r} = \mathbf{v}\cdot\nabla r$. In terms of the physical variables, Eq. (13) may be written as

$$\begin{aligned}\boldsymbol{\epsilon}\cdot\mathbf{J}_0 &+ \overline{\langle s_1 v_{1r}\rangle}\rho_0 T_0'\\ &= -\eta\overline{\langle|\mathbf{J}_1|\rangle}^2 - \mu_\perp\overline{\langle|\mathbf{w}_1|\rangle}^2 - (1/T_0)(\kappa_\parallel\overline{\langle|\nabla_\parallel T_1|^2\rangle}\\ &\quad + \kappa_\perp\overline{\langle|\nabla_\perp T_1|^2\rangle}),\end{aligned} \tag{14}$$

which corresponds to Eq. (3) with $f=0$. Here, $\mathbf{w}_1 = \nabla\times\mathbf{v}_1$.

In an RFP plasma, it is believed that the dynamo[13] or anomalous electric field $\boldsymbol{\epsilon}$ caused by certain kinds of turbulence plays an important role in sustaining the equilibrium configuration through relaxation processes. It is found[1] that, although the tearing mode induced turbulence is necessary to explain field reversal of RFP plasmas, the resistive g-mode-induced turbulence is partially responsible for the RFP configuration. Therefore, the strong coupling of the anomalous electric field $\boldsymbol{\epsilon}$ with the anomalous heat transport $\langle s_1 v_{1r}\rangle$, given in Eq. (14) for the resistive g modes, suggests that there always exists an anomalous heat loss in an RFP plasma.

Equation (14) is considerably simplified for a tokamak plasma. Since $\tilde{s} = \tilde{p}/(\gamma-1)$ for a tokamak case, we have

$$p_0\langle s_1 v_{1r}\rangle = -B_\theta B^2\sigma^2\left\langle\tilde{p}\frac{\partial\tilde{\phi}}{\partial y}\right\rangle[(\gamma-1)\sqrt{\rho_0}]^{-1},$$

$$\boldsymbol{\epsilon}\cdot\mathbf{J}_0 = p_0' B_\theta\sigma^2\left\langle\tilde{v}\frac{\partial\widetilde{A}}{\partial y}\right\rangle(2\sqrt{\rho_0})^{-1},$$

to the lowest order of the inverse aspect ratio ϵ_a. Therefore, it is clear that the radial component of the anomalous heat flux $p_0\langle s_1 v_{1r}\rangle$ is of order ϵ_a^2, which is of the same order as the collisional heat transport $\kappa_\perp T_0'$, while $\boldsymbol{\epsilon}\cdot\mathbf{J}_0$ is of the order ϵ_a^3. Under this ordering in ϵ_a, Eq. (13) becomes

$$-D\left\langle\tilde{p}\frac{\partial\tilde{\phi}}{\partial y}\right\rangle = \chi\overline{\langle|\widetilde{\nabla}_\perp\tilde{p}|^2\rangle}, \tag{15}$$

where $\kappa_\parallel$ is ignored for simplicity. We may also obtain Eq. (15) by multiplying Eq. (10c) by $\tilde{p}$ and taking its $\langle\ \rangle$ average

$$\frac{1}{2}\frac{\partial}{\partial\tau}\langle|\tilde{p}|^2\rangle = -D\left\langle\tilde{p}\frac{\partial\tilde{\phi}}{\partial y}\right\rangle - \chi\langle|\widetilde{\nabla}_\perp\tilde{p}|^2\rangle. \tag{16}$$

Taking the time average of this equation yields Eq. (15), or in terms of the physical variables,

$$-p_0 T_0'\langle s_1 v_{1r}\rangle = (\kappa_\perp/T_0)\langle|\nabla_\perp T_1|^2\rangle. \tag{17}$$

The anomalous electric field ϵ caused by the resistive g modes is thus found to be small and decoupled from the anomalous heat transport in tokamak plasmas. This is consistent with tokamak experiments, noting that ϵ caused by the tearing mode turbulence is known to have a small effect on an enhancement of the classical resistivity.[14] It is also seen from Eq. (17) that $T_0'\langle s_1 v_{1r}\rangle < 0$, or the anomalous heat flux $p_0\langle s_1 v_{1r}\rangle$ across the magnetic surface may be written as $-p_0\chi_{\rm eff}T_0'$, where $\chi_{\rm eff}$ is a non-negative function of such mean quantities as T_0'. Therefore, the anomalous heat transport behaves like the collisional heat conduction in the sense that they both transport heat in the direction opposite to the temperature gradient.

V. BIFURCATION ANALYSIS OF A TOKAMAK PLASMA

In this section, we will derive the dependence of the anomalous heat transport caused by the resistive g modes of Eqs. (10) on the parameter D or the pressure gradient, assuming that Eqs. (10) have steady saturated solutions. Such nonlinear behavior of the solutions of Eqs. (10) is discussed in more detail in Appendix B. It is further assumed for simplicity that mode rational surfaces in a plasma are well separated from each other and mode–mode interactions between two different magnetic surfaces are ignorable. This is certainly not the case for a turbulent plasma, but it is also a reasonable assumption for a certain region of a relatively quiescent plasma. We will, therefore, consider nonlinear modes on a single rational surface, where the modes with the same helicity interact with each other nonlinearly. These modes on a single rational surface can be described as the $\tilde z$-independent solutions of the nonlinear system (10), where $\tilde x$, $\tilde y$, and $\tilde z$ are the coordinates defined in Sec. IV. We will use this coordinate system $(\tilde x,\tilde y,\tilde z)$ throughout this section.

The $\tilde z$-independent normal mode solution of the linearized system of Eqs. (10) have the forms

$$\begin{aligned}\tilde A(\tilde x,\tilde y) &= \tilde A(\tilde x)\exp(imk\tilde y),\\ \tilde\phi(\tilde x,\tilde y) &= \tilde\phi(\tilde x)\exp(imk\tilde y),\\ \tilde p(\tilde x,\tilde y) &= \tilde p(\tilde x)\exp(imk\tilde y).\end{aligned} \tag{18}$$

Here, m is an integer and $k=\pi/\delta_y$ (δ_y is defined in Sec. IV). The helicity of these modes is given by the pitch length $\mu = rB_z/B_\theta$, and we have $\tilde y \propto (z-\mu\theta)$ to the lowest order, where θ and z are the azimuthal angle and the longitudinal coordinate of the cylindrical plasma, respectively. Since δ_y is a quantity of the order of $1/\sqrt{R}$, the modes given in Eqs. (18) with m of order 1 are modes with short wavelengths.

We now consider steady solutions on a single rational surface. When the parameter D, which is the free-energy source of the modes, is slightly larger than the linear stability limit D_L, a perturbation growing from an infinitesimal initial value is expected to saturate with small amplitude. In this case, we are able to obtain expressions of the nonlinearly saturated modes and to estimate the anomalous heat transport by expanding the solutions in terms of the small amplitude.[8]

Let ϵ^2 be the energy of the perpendicular flow of such a solution, i.e.,

$$\epsilon^2 = \tfrac{1}{2}\langle|\tilde\nabla_\perp\tilde\phi|^2\rangle. \tag{19}$$

Assuming that ϵ is a small number, we introduce new dependent variables $\hat A$, $\hat\phi$, and $\hat p$ defined by $\tilde A = \epsilon\hat A$, $\tilde\phi = \epsilon\hat\phi$, and $\tilde p = \epsilon\hat p$. Since the steady $\tilde z$-independent solutions are considered, we set $\partial/\partial\tau = 0$ and $\partial/\partial\tilde z = 0$ (i.e., $\partial/\partial\tilde\theta = \tilde x\partial/\partial\tilde y$). Therefore, Eqs. (10) become

$$\epsilon\{\hat\phi,\hat A\} = \tilde x\frac{\partial\hat\phi}{\partial\tilde y} + R^{-1}\tilde\Delta_\perp\hat A, \tag{20a}$$

$$\epsilon\{\hat\phi,\tilde\Delta_\perp\hat\phi\} = \tilde x\frac{\partial\tilde\Delta_\perp\hat A}{\partial\tilde y} + \epsilon\{\hat A,\tilde\Delta_\perp\hat A\} - \frac{\partial\hat p}{\partial\tilde y} + M\tilde\Delta_\perp^2\hat\phi, \tag{20b}$$

$$\varepsilon\{\hat\phi,\hat p\} = -D\frac{\partial\hat\phi}{\partial\tilde y} + \chi\tilde\Delta_\perp\hat p. \tag{20c}$$

Our aim in this section is to solve the system (20), together with Eq. (19), assuming that ϵ is a small quantity.

In the system (20), however, the existence of the nontrivial solutions is not guaranteed for all ϵ (with the fixed parameters D, R, M, and χ). In fact, if $\epsilon = 0$, then the system (20) forms a linear eigenvalue problem and that $D = D_L$ is the condition that the nontrivial solutions exist. Thus, for a finite ϵ, we choose a proper $D = D(\epsilon)$ as a function of ϵ so that we expect the solutions to exist for any (small) ϵ. Here we assume the analytic dependence of the solutions and D on ϵ and expand the solutions and D in power series of ϵ as follows:

$$\hat A = \sum_{n=0}^{\infty}\hat A_n\epsilon^n,\quad \hat\phi = \sum_{n=0}^{\infty}\hat\phi_n\epsilon^n,\quad \hat p = \sum_{n=0}^{\infty}\hat p_n\epsilon^n,$$

and

$$D = D_L + \sum_{n=1}^{\infty}D_n\epsilon^n.$$

We now seek formal solutions of this type by substituting these quantities into Eqs. (20). Considering the case where $D_L > 0$ (this choice is always possible, as shown in Appendix B), we obtain the following system as the zeroth-order system in ϵ

$$L\begin{pmatrix}\hat\phi_0\\ \hat p_0\end{pmatrix} \equiv \begin{pmatrix}-R\tilde x^2\partial^2\hat\phi_0/\partial_y^2 - \partial\hat p_0/J\tilde y + M\tilde\Delta_\perp^2\hat\phi_0\\ \partial\hat\phi_0/\partial\tilde y - (\chi/D_L)\tilde\Delta_\perp\hat p_0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \tag{21}$$

and $\hat A_0$ is determined from the equation $\tilde\Delta_\perp\hat A_0 = -R\tilde x\partial\hat\phi_0/\partial\tilde y$. The lowest-order term of Eq. (19) determines the amplitude of $\hat\phi_0$: $\langle|\tilde\nabla_\perp\hat\phi_0|^2\rangle = 2$.

We now consider marginally stable solutions having the following forms: $\hat A_0 = \hat A_{01}(\tilde x)\cos k\tilde y$, $\hat\phi_0 = \hat\phi_{01}(\tilde x)\sin k\tilde y$, and $\hat p_0 = \hat p_{01}(\tilde x)\cos k\tilde y$, where k is a given wavenumber. In this case, Eq. (21) becomes

$$L_k\begin{pmatrix}\hat\phi_{01}\\ \hat p_{01}\end{pmatrix} \equiv \begin{pmatrix} Rk^2\tilde x^2 + M(\partial_x^2 - k^2)^2 & k \\ k & -(\chi/D_L)(\partial_x^2 - k^2)\end{pmatrix} \times \begin{pmatrix}\hat\phi_{01}\\ \hat p_{01}\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}. \tag{22}$$

Here, $\partial_x = \partial/\partial\tilde x$, and the functions $\hat\phi_{01}$ and $\hat p_{01}$ are taken to be real-valued functions of $\tilde x$. We think of L_k as a linear operator applied to a pair of real-valued functions $(u_1,u_2)^T$ of $\tilde x$, where superscript T denotes the transpose of the vector and u_1 and u_2 are sufficiently smooth square-integrable functions. We define the inner product of such functions $(u_1,u_2)^T$ and $(v_1,v_2)^T$ as $(u_1,u_2)(v_1,v_2)^T = \langle u_1v_1 + u_2v_2\rangle$. It is clear that L_k is a Hermitian operator with respect to this inner product.

In Eq. (22), we note that the linear stability limit D_L is a function of k. In order not to generalize the problem too much, we here choose the wavenumber k such that the modes with this k are the only marginal modes; in other words, all the other modes whose $\tilde y$ dependence is given by either $\sin lk\tilde y$ or $\cos lk\tilde y$ $(l\neq 1)$ have negative linear growth rate when $D = D_L$. In this case, the operator L_{lk}, where l is an integer, is always invertible unless $l=1$, so that the linear equation $L_{lk}(u_1,u_2)^T = (0,0)^T$ only gives the trivial solution if $l\neq 1$.

With these solutions of the zeroth-order system, the first-order system in ϵ becomes

$$\langle \tilde\nabla_\perp\hat\phi_0\cdot\nabla_\perp\hat\phi_1\rangle = 0, \tag{23}$$

$$L\begin{pmatrix}\hat\phi_1\\ \hat p_1\end{pmatrix} = \begin{pmatrix}\{\hat\phi_0,\tilde\Delta_\perp\hat\phi_0\} - R\tilde x\,\partial\{\hat\phi_0,\hat A_0\}/\partial\tilde y - \{\hat A_0,\tilde\Delta_\perp\hat A_0\}\\ -(\{\phi_0,p_0\} + D_1\,\partial\hat\phi_0/\partial\tilde y)/D_L\end{pmatrix}, \tag{24}$$

$$\tilde\Delta_\perp\hat A_1 = -R\tilde x\,\partial\hat\phi_1/\partial\tilde y + R\{\hat\phi_0,\hat A_0\}. \tag{25}$$

If we expand the solutions $\hat A_1$, $\hat\phi_1$, and $\hat p_1$ in Fourier series in $\tilde y$ such as $\hat A_1 = \Sigma_l\,\hat A_{1l}\cos lk\tilde y$, $\hat\phi_1 = \Sigma_l\,\hat\phi_{1l}\sin lk\tilde y$, and $\hat p_1 = \Sigma_l\,\hat p_{1l}\cos lk\tilde y$, it is clear that we obtain only three sets of equations for $l = 0$, 1, and 2 from Eqs. (23)–(25).

For $l=1$, we have

$$L_k\begin{pmatrix}\hat\phi_{11}\\ \hat p_{11}\end{pmatrix} = \begin{pmatrix}0\\ -k(D_1/D_L)\hat\phi_{01}(\tilde x)\end{pmatrix}. \tag{26}$$

The necessary and sufficient condition that Eq. (26) have a nontrivial solution is that the right-hand side is orthogonal to the null space of the adjoint operator L_k^+ of L_k. This condition determines D_1. Since $L_k = L_k^+$, we only have to require that the inner product of $(\hat\phi_{01},\hat p_{01})$ and the right-hand side of Eq. (26) be equal to zero

$$k(D_1/D_L)\langle\hat p_{01}\,\hat\phi_{01}\rangle = 0. \tag{27}$$

Multiplying the second equation of Eq. (22) by $\hat p_{01}$ and taking the $\langle\ \rangle$ average of the resulting equation yields

$$k\langle\hat p_{01}\hat\phi_{01}\rangle = -(\chi/D_L)\langle(\partial_x\hat p_{01})^2 + k^2p_{01}^2\rangle,$$

so $\langle\hat p_{01}\,\hat\phi_{01}\rangle$ is nonzero for nonzero $\hat p_{01}$. Therefore, from Eq. (27), $D_1 = 0$ and the solution of Eq. (26) becomes $C(\hat\phi_{01},\hat p_{01})^T$ with some constant C. However, from Eq. (23), we have $\langle\tilde\nabla_\perp\hat\phi_{01}\cdot\tilde\nabla_\perp\hat\phi_{11}\rangle = 0$, so $C=0$. For $l=0$ and 2, we obtain $(\hat\phi_{10},\hat p_{10})^T$ and $(\hat\phi_{12},\hat p_{12})^T$ from Eq. (23) by just inverting the operators L_0 and L_{2k}.

The second-order system in ϵ from Eqs. (20) is

$$L\begin{pmatrix}\hat\phi_2\\ \hat p_2\end{pmatrix} = \begin{pmatrix}f\\ -(g + D_2\partial\hat\phi_0/\partial\tilde y)/D_L\end{pmatrix} \tag{28}$$

and $\tilde\Delta_\perp\hat A_2 = -R\tilde x\partial\hat\phi_2/\partial\tilde y + h$, where

$$f = \{\hat\phi_0,\tilde\Delta_\perp\hat\phi_1\} + \{\hat\phi_1,\tilde\Delta_\perp\hat\phi_0\} - \{\hat A_0,\tilde\Delta_\perp\hat A_1\} - \{\hat A_1,\tilde\Delta_\perp\hat A_0\} - x\frac{\partial h}{\partial y},$$

$$g = \{\hat\phi_0,\hat p_1\} + \{\hat\phi_1,\hat p_0\},$$

and

$$h = R(\{\hat\phi_0,\hat A_1\} + \{\hat\phi_1,\hat A_0\}).$$

Equations (28) are solved by using the following Fourier expansion in $\tilde y$: $\hat A_2 = \Sigma_l\,\hat A_{2l}\cos lk\tilde y$, $\hat\phi_2 = \Sigma_l\,\hat\phi_{2l}\sin lk\tilde y$, $\hat p_2 = \Sigma_l\,\hat p_{2l}\cos lk\tilde y$, $f = \Sigma_l\,f_l\sin lk\tilde y$, and $g = \Sigma_l\,g_l\cos lk\tilde y$. For $l=1$, we have

$$L_k\begin{pmatrix}\hat\phi_{21}\\ \hat p_{21}\end{pmatrix} = \begin{pmatrix}f_1\\ -(g_1 + kD_2\hat\phi_{01})/D_L\end{pmatrix}, \tag{29}$$

the solvability condition of which determines D_2, as in the case of Eq. (26). The necessary and sufficient condition that Eq. (29) have a solution is, as before, that the inner product of $(\hat\phi_{01},\hat p_{01})$ and the right-hand side of Eq. (29) be equal to zero. Therefore, D_2 is given by

$$D_2 = (D_L\langle\hat\phi_{01}f_1\rangle - \langle\hat p_{01}\,g_1\rangle)/k\langle\hat p_{01}\hat\phi_{01}\rangle.$$

Assuming that this D_2 is not zero, we are able to relate the anomalous heat transport $\langle s_1v_{1r}\rangle$, which is proportional to $\langle\tilde p\,\partial\tilde\phi/\partial\tilde y\rangle$, to the parameter D. Since $\langle\tilde p\,\partial\tilde\phi/\partial\tilde y\rangle = \epsilon^2k\langle\hat p_{01}\hat\phi_{01}\rangle$ and $D = D_L + \epsilon^2D_2$ to the lowest order of ϵ, we have $\langle\tilde p\,\partial\tilde\phi/\partial\tilde y\rangle = k\langle\hat p_{01}\,\hat\phi_{01}\rangle\,(D - D_L)/D_2$, or

$$p_0\langle s_1v_{1r}\rangle = -\frac{B_\theta B^2\sigma^2}{(\gamma-1)\sqrt{\rho_0}}k\langle\hat p_{01}\hat\phi_{01}\rangle\frac{(D-D_L)}{D_2} \tag{30}$$

near $D = D_L$. Equation (30) gives the dependence of the anomalous heat flux $p_0\langle s_1v_{1r}\rangle$ on the parameter D, which we have been after in this section.

We have thus derived the dependence of the anomalous heat flux on the parameter D. Since this derivation is based on a small amplitude expansion, the result is reliable when D is slightly larger than the linear stability limit D_L. In this region, it is found that the anomalous heat flux is linear with $(D - D_L)$. When the parameter D is much larger than the linear stability limit D_L, the nonlinearity of the system is so strong that we need numerical calculations to estimate the anomalous heat transport. In Sec. VI, we will show some results of the numerical simulations and discuss the validity of the analytical results of Eq. (30).

VI. NUMERICAL CALCULATIONS OF THE REDUCED EQUATIONS

In this section, the results of numerical calculations of the reduced equations are presented. Here we only deal with the system (10) of tokamak plasma, in which the anomalous heat transport is decoupled from the anomalous electric field. It is shown that, in the tokamak plasma, steady convection within a boundary layer is attained and the anomalous heat transport arising from this convection weakly depends on the collisional heat conductivity. Therefore, when the effect of the collisional heat conduction is small, the total heat transport is dominated by the anomalous one caused by the convective cells. It is also shown that the analytical estimate of the anomalous heat transport given by Eq. (30) agrees well with the results of the numerical calculations near the linear stability limit D_L. Another important result obtained in this section is an observation of the secondary bifurcation in three-dimensional (3-D) calculations of the system, which occurs when the second higher modes become linearly unstable. For the system (8) of an RFP plasma, the nonlinear solutions have more complicated features. The detailed numerical study of this system will be presented elsewhere.

The nonlinear initial value code to solve the reduced equations (10) is developed from the HIB code.[15,16] The equations are solved in the domain of the slab geometry, i.e., $|\tilde{x}|\leqslant\delta_x, |\tilde{y}|\leqslant\delta_y$, and $0\leqslant\tilde{z}\leqslant L_z$, which corresponds to a boundary layer in the plasma. For the boundary conditions, all the physical quantities are assumed to vanish at $|\tilde{x}| = \delta_x$ and to be periodic in $\tilde{y}$ and $\tilde{z}$, as in Sec. IV. The numerical method used in the code is a finite difference scheme for the variable $\tilde{x}$, Fourier component representations for the variables $\tilde{y}$ and $\tilde{z}$, and the predictor–corrector method for time τ. For simplicity, as in Sec. V, only the Fourier cosine components for $\tilde{\phi}$ and $\tilde{p}$ and the Fourier sine components of $\tilde{A}$ are taken into consideration.

We now take the small scale parameter δ introduced in Sec. III to be $1/\sqrt{R}$. In terms of this δ, the size of the domain used for our boundary conditions is taken to be $\delta_x/\delta = 25$ and $\delta_y/\delta = \pi$ and $L_z/r_0 = 2.5\pi$. It is confirmed numerically that the solutions decay rapidly as $|\tilde{x}|\rightarrow\delta_x$ and, moreover, these solutions hardly depend on the choice of δ_x if δ_x/δ is taken to be 25 or larger. Therefore, we choose $\Delta = \delta_y$ for the definition of the averaging operator $\langle\ \rangle$; in fact, it is observed in these numerical computations that the typical mode width in the $\tilde{x}$ direction Δ is of the same order of the one in the $\tilde{y}$ direction δ_y.

The boundary conditions imposed here depend on the magnetic Reynolds number R since δ_x and δ_y are taken to be proportional to $\delta = 1/\sqrt{R}$. However, after the scale transformation (9) is applied to this system, this dependence of the boundary conditions disappears. The solutions of this transformed system thus depend only on the parameters D, M_R, and K_R, but not on R. In actual numerical calculations, therefore, we solve this transformed system and leave the magnetic Reynolds number R as an undetermined parameter.

Typically we use the following values of the parameters in our calculations unless otherwise specified: $D = 0.20$ and $M_R, K_R = 1 \sim 10^{-2}$. Here, we note that $D < 0.25$ is the ideal stability condition by Suydam.[11] We also note that $M_R \sim K_R \sim \beta\sqrt{m_i/m_e}$, where β is the plasma beta, m_i and m_e are the masses of the ion and the electron, respectively, in a collisional plasma given in Ref. 9. As before, the parallel heat conductivity $\chi_\parallel$ is taken to be smaller than $O(1)$ for simplicity, so that it does not enter the equations. However, we point out that some preliminary calculations including $\chi_\parallel$ show that $\chi_\parallel$ has a stabilizing effect, and thus reduces somewhat the anomalous heat transport across the magnetic surface. In most runs presented here, 150 mesh points in the $\tilde{x}$ direction and seven modes ($0\leqslant m\leqslant 6$) in the $\tilde{y}$ direction and five modes ($-2\leqslant n\leqslant 2$) in the $\tilde{z}$ direction are employed, where m and n are defined by the relations $\partial/\partial\tilde{y} = \pi m/\delta y$ and $\partial/\partial\tilde{z} = 2\pi n/L_z$, respectively. It is confirmed by varying the number of the mesh points and the number of the modes that this is a sufficient numerical resolution to obtain the correct saturation levels of the convection. The initial condition given to the calculations is a sufficiently small perturbation of the $m = 1$ component of $\tilde{\phi}$.

We now present the anomalous heat transport obtained from 2-D numerical calculations, where the $\tilde{z}$ dependence of the solutions is ignored ($n = 0$). The anomalous heat transport across the magnetic surface can be measured by the Nusselt number

$$\mathrm{Nu} = 1 + p_0\langle s_1 v_{1r}\rangle/(-\kappa_\perp T_0').$$

This is the ratio of the actual heat flux, i.e., the sum of the collisional heat flux $-\kappa_\perp T_0'$ and the anomalous heat flux $p_0\langle s_1 v_{1r}\rangle$ due to the convection, to the collisional heat flux $-\kappa_\perp T_0'$. We note that the Nusselt number Nu is invariant under the scale transformation (19), so that Nu also depends only on D, M_R, and K_R. Figure 1(a) shows the time evolution of the Nusselt number Nu. It is seen that the modes grow linearly at the initial stage and reach the steady convection state after about $100\tau_A$, where τ_A is the Alfvén time defined as $r_0\sqrt{\rho_0}/(B_\theta|\sigma|)$ [Eqs. (5)]. The contours of the streamfunction $\tilde{\phi}$ and the mode structure of $\tilde{p}$ at $\tau = 300\ \tau_A$ are shown in Figs. 1(b) and 1(c), respectively. The saturation mechanism is a quasilinear stabilization or a flattening of the mean pressure gradient, which appears as the dominance of the $m = 0$ mode of the perturbed pressure $\tilde{p}$.[17] (Although the parameter D is taken to be a constant during computations, the $m = 0$ mode of $\tilde{p}$ acts as the *time-dependent* part of the mean pressure.) Figures 2 show the dependence of the Nusselt number Nu on the parameter D obtained from the theory [Eq. (30)] and the numerical simulations. Here, the theoretical value of Nu is calculated from the following expression:

$$\mathrm{Nu} = 1 - \frac{2}{\gamma}\frac{k\langle\hat{p}_{01}\hat{\phi}_{01}\rangle}{K_\perp D_2}\frac{(D - D_L)}{D_L}, \tag{31}$$

where we set $1/D = 1/D_L$ since $|D - D_L| \ll D_L$ was assumed to derive Eq. (30). In order to evaluate the coefficient of $(D - D_L)$ in Eq. (31), the linear equations (21) and (24) are solved numerically. It is seen that these Nusselt numbers obtained from the bifurcation analysis are in good agreement with the results of the nonlinear simulations near the critical point D_L. We also note that the Nusselt number Nu is smaller in the more heat conductive case (b) than in the less heat conductive case (a).

We now seek the dependence of the Nusselt number Nu on the normalized diffusion coefficients M_R and K_R for a

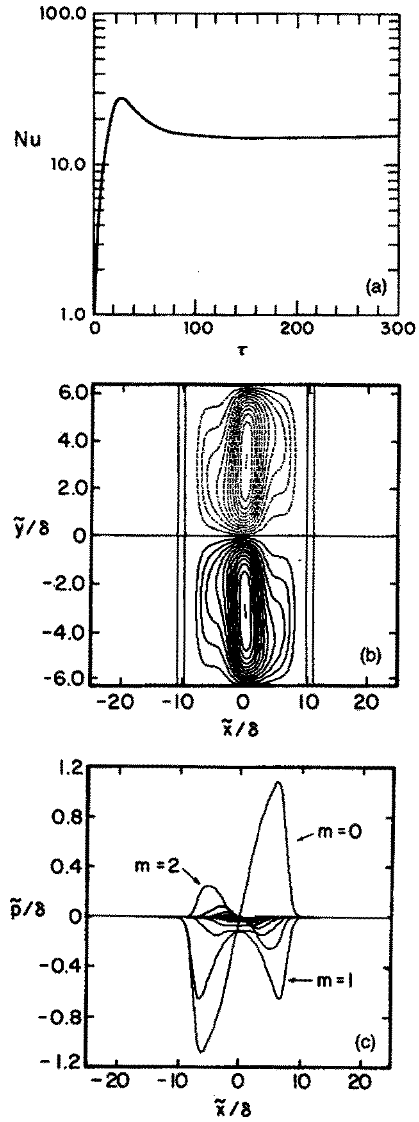

FIG. 1. Formation of convective cells by the resistive g modes. (a) Time evolution of the Nusselt number Nu. (b) The contours of the streamfunction $\tilde{\phi}$ and (c) the mode structure of $\tilde{p}$ at $\tau = 300\,\tau_A$. The solid contour lines in (b) indicate the positive values of $\tilde{\phi}$ and the dotted contour lines indicate its negative values. Here, 12 modes are included, $M_R = 1$, $K_R = 5\times 10^{-3}$, and $D = 0.2$.

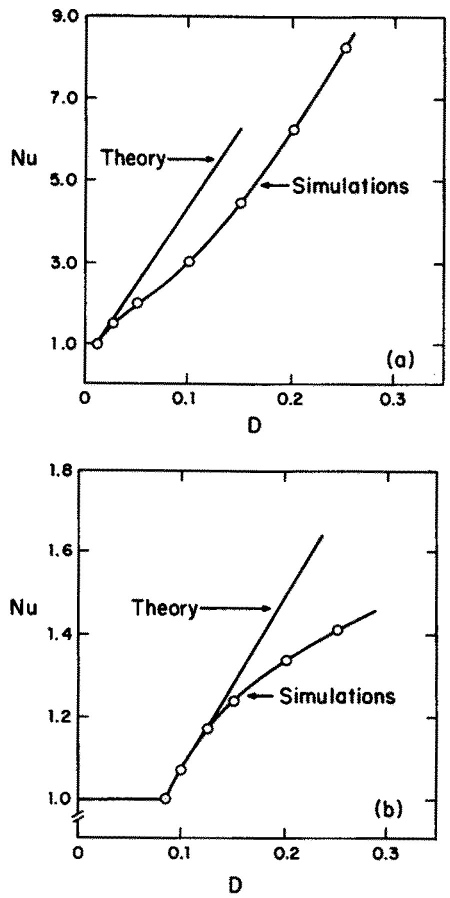

FIG. 2. The Nusselt numbers Nu as functions of D obtained from the theory [Eq. (30)] and the simulations of system (10). Here, $M_R = 1$, $K_R = 10^{-2}$ for (a) and $K_R = 10^{-1}$ for (b).

fixed D. Figure 3(a) shows that the anomalous heat flux $p_0\langle s_1 v_{1r}\rangle$ varies weakly with normalized heat conductivity K_R. Here, the nondimensional quantity Ja denotes the anomalous heat flux $p_0\langle s_1 v_{1r}\rangle$ measured by the quantity $-\eta\rho_0 T_0'$, i.e., $Ja = -p_0\langle s_1 v_{1r}\rangle/\eta\rho_0 T_0'$. This weak dependence of Ja on K_R accounts for the almost linear dependence of the Nusselt number Nu on K_R^{-1}. In a realistic parameter range for fusion experiments such that $D = 0.2$ and $M_R, K_R = 1.0 \sim 10^{-2}$, it is seen that the Nusselt number Nu varies from 1 to 10; in other words, the steady convection enhances the heat transport up to one order of magnitude over the collisional heat conduction. It is also seen that when K_R is less than 10^{-2}, the anomalous heat flux or the saturation level of the modes increases as some power of K_R^{-1}. This corresponds to the fact that there is no saturated mode nor stationary turbulence when $K_R = 0$, which was stated and proved as Proposition 2 in Appendix B. Figure 3(b) shows that the anomalous heat flux represented by the Nusselt number Nu with fixed K_R also depends weakly on the normalized viscosity M_R.

We now proceed to 3-D calculations of the system (10), where the $\tilde{z}$ dependence of the modes or interactions of modes localized on different rational surfaces are taken into account. Since the $\tilde{z}$ derivatives of the system (10) appear only as the form $\tilde{x}\,\partial/\partial\tilde{y} + \partial\tilde{z}$, which can be written as $i(\tilde{x}\pi m/\delta_y + 2\pi n/L_z) = i(\pi m/\delta_y)(\tilde{x} + 2n\delta_y/mL_z)$ for the linear solutions, the linear (m,n) mode becomes identical to the linear $(m,0)$ mode by the transformation $\tilde{x} + 2n\delta_y/mL_z \to \tilde{x}$; in other words, the linear (m,n) mode is localized at $x = -2n\delta_y/mL_z$ and has the same structure and the growth rate as the linear $(m,0)$ mode. In 3-D calculations of the plasma where mode rational surfaces are well separated, if only the $m = 1$ modes are linearly unstable, those modes saturate quasilinearly and the 2-D dynamics accounts for each convection cell [Fig. 4(a)]. For the plasma with the parameter D such that the $m = 2$ modes are also linearly unstable, however, the 2-D and 3-D nonlinear analyses of the modes exhibit significantly different results. In the 2-D case, the $m = 2$ mode localized at $\tilde{x} = 0$ is stabilized by the flattened pressure gradient (the constant mean pres-

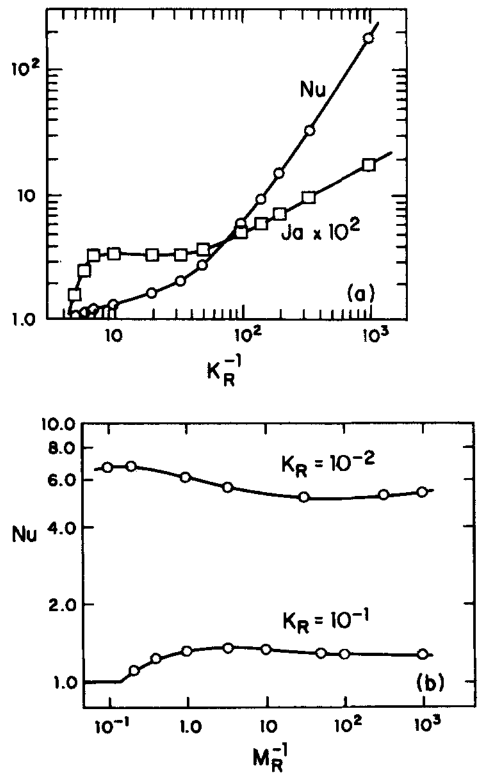

FIG. 3. The dependence of the anomalous heat transport on the diffusivity. (a) The Nusselt number Nu (○) and the anomalous heat flux Ja (□) as functions of the inverse of normalized heat conductivity K_R^{-1} and (b) the dependence of the Nusselt number Nu on the inverse of the normalized viscosity M_R^{-1}.

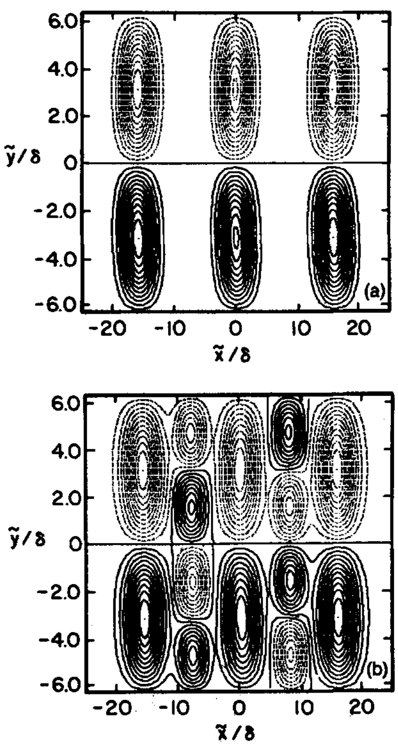

FIG. 4. The contours of the streamfunction $\tilde{\phi}$ obtained from three-dimensional simulations (a) at $D = 0.15$ and (b) at $D = 0.23$. Here, $M_R = 1$ and $K_R = 6.0\times 10^{-2}$. The solid contour lines indicate the positive values of $\tilde{\phi}$ and the dotted contour lines indicate its negative values.

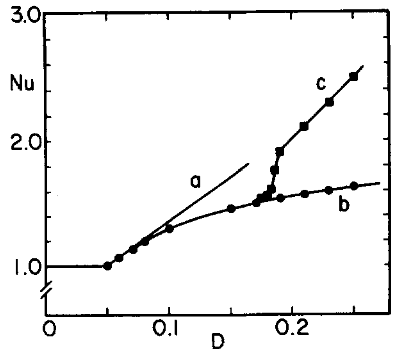

FIG. 5. The Nusselt numbers Nu as functions of D obtained from (a) the theory [Eq. (30)], (b) the 2-D simulations of a single rational surface (denoted by ●), and (c) the 3-D simulations of multirational surfaces (denoted by ■). Here, $M_R = 1.0$ and $K_R = 6.0\times 10^{-2}$, i.e., a highly diffusive case, so Nu is relatively small but the secondary bifurcation is clearly observed.

sure gradient $D +$ the gradient of the $m = 0$ component of $\tilde{p}$), mainly caused by the saturated $m = 1$ mode, as shown in Fig. 1(c). On the other hand, in the 3-D case, for example, the $(m = 1, n = 0)$ and the $(m = 1, n = 1)$ modes create the $(m = 2, n = 1)$ mode, which is localized at $\tilde{x} = -y_0/2z_0$ or just between these $m = 1$ modes [Fig. 4(b)]. In the region between the dominant $m = 1$ modes, the mean pressure is not sufficiently flattened by the $m = 1$ modes, and those $m = 2$ modes grow until they saturate by their own quasilinear effects. These $m = 2$ modes have relatively large amplitude so that they enhance the anomalous heat transport $\langle s_1 v_{1r} \rangle$. Figure 5 shows the Nusselt number as functions of D, comparing the analytical estimate (a), the 2-D nonlinear simulations (b), and the 3-D nonlinear simulations (c), where the averaging $\langle\ \rangle$ is taken over a single dominant cell. It is shown that the 2-D calculations of single helicity modes and the 3-D calculations of multihelicity modes exhibit the same value of Nu for $D_L = 0.052 \leqslant D \leqslant 0.18$, where the well-separated steady convection cells of the $m = 1$ modes are observed in the 3-D calculations [Fig. 4(a)]. The 3-D calculations, however, give rise to a clear secondary bifurcation at D slightly less than $D_{L2} = 0.19$, at which the $m = 2$ mode becomes linearly unstable while the 2-D calculations show no drastic change of Nu as D exceeds D_{L2}. This indicates that, if $D \geqslant D_{L2}$, the $\tilde{z}$-independent solutions of the system (10), represented by the curb (b) in Fig. 5, is not stable anymore three dimensionally and cannot be observed in reality. No tertiary bifurcation is observed since the plasma treated in this example is so dissipative that all the modes with $m \geqslant 3$ are stable for $D \leqslant 0.25$ or under the ideal stability condition. For a less diffusive case, however, it is numerically observed that more high-m modes are destabilized with smaller values of D, a series of bifurcation increases the Nusselt number Nu, and the system approaches fully developed turbulence. Such fully developed turbulence caused by the resistive g modes is examined by Carreras *et al.*[18] based on equations similar to Eqs. (10), and it is shown that the anomalous heat flux is larger by almost two orders of magnitude than the collisional heat transport. [It is easy to check that the quantity $\tau_R D_{xx}/a^2$ in Ref. 18 is related to the anomalous heat flux Ja in Fig. 3(a) in this paper by

$\tau_R D_{xx}/a^2 = (\gamma - 1)Ja$.] Our numerical calculations also show such results for less diffusive plasmas: For example, for $M_R = 1.0$, $K_R = 1.0\times 10^{-2}$, and $D = 0.25$, the time average of the Nusselt number Nu $\simeq 70$. The three-dimensional numerical simulations presented here thus indicate that a series of such bifurcations, as shown in Fig. 5, leads the system from coherent convection cells to fully developed turbulence, although the detailed structure of the higher-order bifurcations (such as the Hopf bifurcation[19]) is not clearly identified in the present work.

VII. DISCUSSION AND CONCLUSIONS

In this paper, we have discussed the anomalous heat transport caused by the resistive g-mode fluctuations in a plasma. The two systems of nonlinear equations describing such fluctuations have been derived, one for an RFP plasma with a possible high-beta value and the other for a cylindrical tokamak plasma, under the assumption that the fluctuating quantities have much smaller scales in space and time than the corresponding mean quantities. In these equations, the effects of all the collisional diffusion coefficients, i.e., resistivity, viscosity, and heat conductivity, have been taken into account.

The inclusion of all the collisional diffusion coefficients in the reduced equations allows us to derive relations between different types of anomalous transport. It is shown based on these reduced equations that the anomalous heat transport $\langle s_1 \mathbf{v}_1 \rangle$ is related to the anomalous electric field $\boldsymbol{\epsilon}$ in an RFP plasma through Eq. (14). Since $\boldsymbol{\epsilon}$ caused by the resistive g modes is considered to play a partial role in dynamo activity of an RFP,[14] it is expected that there is a large anomalous heat loss when there is a strong dynamo activity in an RFP. On the other hand, it is also shown that, in a tokamak, the anomalous electric field is smaller than the anomalous heat transport by an order of the inverse aspect ratio. In this case, the anomalous heat flux $p_0\langle s_1 v_{1r}\rangle$ can be written as $-\rho\chi_{\rm eff}\, T_0'$ with a non-negative function $\chi_{\rm eff}$.

As observed in the numerical simulations presented in Sec. VI, the small perturbations that grow linearly in the beginning saturate with finite amplitude and form steady convection cells on the rational surface. By regarding this steady convection as a bifurcation from the null solutions of the reduced equations, we have applied the nonlinear bifurcation analysis to the system (10) in order to derive the dependence of the anomalous heat transport on the mean pressure gradient or the parameter D. In this method, the reduced equations are expanded in terms of the small amplitude, and the complete algorithm to determine all the higher-order terms is obtained. It is shown by this method that, to the lowest order, the anomalous heat transport is proportional to the difference of the parameter D and its linear stability limit D_L, that is, $(D - D_L)$.

The validity of the analysis mentioned above is confirmed by using the numerical calculations. These numerical calculations also determine the dependence of the anomalous heat transport on various parameters in the equations. It is found numerically that: (1) the anomalous heat flux obtained from the bifurcation analysis in Sec. V is in good agreement with the one obtained from the numerical simulations near the linear stability limit D_L; (2) the anomalous heat flux varies weakly with the normalized diffusion coefficients K_R and M_R; and (3) the Nusselt number Nu, which is the ratio of the total heat flux to the collisional heat flux, therefore becomes significantly large when the collisional heat conduction is small. In a realistic parameter range of fusion experiments such that $D = 0.2$ and $M_R, K_R = 1.0 \sim 10^{-2}$, the Nusselt number Nu obtained from the 2-D calculations varies from 1 to 10. In other words, the steady convection on each rational surface enhances the heat transport up to one order of the magnitude over the collisional heat conduction. In the 3-D calculations, we obtain evidence that a series of higher-order bifurcations leads to fully developed turbulence and the associated anomalous heat transport increases up to almost two orders of magnitude over the collisional heat transport or Nu $\lesssim 10^2$, which is consistent with the result of Ref. 18.

From the analyses in Secs. IV–VI, we are able to derive a scaling law of the anomalous heat conductivity arising from resistive g modes. From Eq. (31), the Nusselt number Nu is given by Nu $- 1 = \alpha_N (D - D_L)$, with some constant α_N if $D \geqslant D_L$ and Nu $= 1$ if $D \leqslant D_L$. Defining the effective heat conductivity $\chi_{\rm eff}$ by $p_0\langle s_1 v_{1r}\rangle = -\rho_0\chi_{\rm eff}\, T_0'$ as before, we can write $\chi_{\rm eff} = \alpha_N K_R \eta (D - D_L)/(\gamma - 1)$. From the numerical calculations in Sec. VI, it is found that the anomalous heat flux weakly depends on the diffusion coefficients K_R and M_R, and so the numerically obtained value of $\alpha_N K_R \simeq 0.2 \sim 0.4$ in a wide range of parameters. Therefore, defining the critical pressure gradient p_c' by $D_L = -2rp_c'/B^2\sigma^2$, we have

$$\chi_{\rm eff} = \eta[-\alpha(p_0' - p_c')/rB_z^2(q'/q)^2], \tag{32}$$

where $\alpha = 2\alpha_N K_R/(\gamma - 1) \simeq 0.4 \sim 0.8$ if $-p_0' \geqslant -p_c'$. We note that this scaling of $\chi_{\rm eff}$ is similar to the energy transport coefficient D_0 of low-beta RFP plasmas obtained for the resistive g-mode turbulence by Bhattacharjee and Hameiri,[12] except for the dependence on the critical pressure gradient p_c' in Eq. (32). We also note that Eq. (32) is similar to the leading scaling of the anomalous heat conductivity D_{xx} obtained by Carreras *et al.*,[18] again except for p_c', noting the correspondence between $B_\theta^{-2} d\Omega/dr$ in Ref. 18 and $1/rB_z^2$ in our notation. As the normalized diffusion coefficients M_R and K_R become small, however, the plasma becomes more linearly unstable and the critical value of $-p_c'$ decreases. In this limit, where the final saturated state is more likely to be turbulence than steady convection, the scaling of $\chi_{\rm eff}$ of Eq. (32) agrees with D_0 of Ref. 12 and the leading scaling of D_{xx} of Ref. 18.

In this paper, up to Sec. IV, we develop a theory that is applicable to both fully developed turbulence and some coherent motion, such as steady convection. However, in Secs. V and VI, where we actually estimate the anomalous heat transport, we only treat the steady convection on each rational surface. This coherent structure of the modes considerably simplifies the physical model and allows a mathematically rigorous treatment of the system. The scaling of $\chi_{\rm eff}$ of Eq. (32) is also derived under this condition so that it is expected to give a good estimate of the anomalous heat transport in a relatively quiescent plasma where the steady convection persists rather than developing into turbulence. It is interesting, however, that $\chi_{\rm eff}$ obtained under this condition still agrees with at least the leading scaling of the anomalous transport[12,18] caused by fully developed turbulence.

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APPENDIX A: REDUCED EQUATIONS WITH PARALLEL VISCOSITY

Taking into account the effect of the parallel viscosity $\mu_\parallel$, we will derive the nonlinear reduced equations of the resistive g-mode fluctuations. In addition to the assumptions made in Sec. III, we assume that $\mu_\parallel = O(1)$. We also assume that $\nabla_\perp \cdot \mathbf{v}_1 = O(\delta)$, as in Sec. III, so that

$$\lambda_1 = (1/\mathbf{B}_0^2)\mathbf{B}_0\cdot[(\mathbf{B}_0 + \mathbf{B}_{1\perp})\cdot\nabla\mathbf{v}_1] - \tfrac{1}{3}\nabla_\perp\cdot\mathbf{v}_1$$

and

$$-(\nabla\cdot\Pi)_1 = 3\mu_\parallel \mathbf{B}_0(\mathbf{B}_0 + \mathbf{B}_{1\perp})\cdot\nabla(\lambda_1/B_0^2) - \mu_\parallel \nabla\lambda_1 + \mu_\perp\Delta_\perp \mathbf{v}_1.$$

Therefore, the perpendicular component of the momentum equation becomes $\nabla_\parallel(p_1 + \mathbf{B}_0\cdot\mathbf{B}_1 - 3\mu_\parallel\lambda_1) = 0$ to $O(1)$, which leads to $b_\parallel \equiv -\mathbf{B}_0\cdot\mathbf{B}_1 = p_1 - 3\mu_\parallel\lambda_1$. Equations (4b) and (4e) are also modified, respectively, to

$$\rho_0 \frac{d}{dt}\Delta_\perp\phi = (\mathbf{B}_0 + \mathbf{B}_{1\perp})\cdot\nabla(\Delta_\perp A) - 2\mathbf{b}\times\nabla(p_0 + \tfrac{1}{2}\mathbf{B}_0^2)\cdot\nabla b_\parallel - 2\mathbf{b}\cdot\nabla(\tfrac{1}{2}\mathbf{B}_0^2)(\nabla_\perp A) - 3\mu_\parallel(\mathbf{J}_0\cdot\mathbf{B}_0)(\mathbf{B}_0 + \mathbf{B}_1)\cdot\nabla(\lambda_1/\mathbf{B}_0^2) + \mu_\perp\Delta_\perp^2\phi \tag{A1}$$

and

$$\rho_0\frac{d}{dt}v = \mathbf{B}_0\cdot\nabla p_1 + \mathbf{B}_{1\perp}\cdot\nabla b_\parallel + \mathbf{B}_1\cdot\nabla p_0 - 3\mu_\parallel \mathbf{B}_0^2(\mathbf{B}_0 + \mathbf{B}_{1\perp})\cdot\nabla(\lambda_1/\mathbf{B}_0^2) + \mu_\parallel\mathbf{B}_0\cdot\nabla\lambda_1 + \mu_\perp\Delta_\perp v, \tag{A2}$$

where λ_1 is given by

$$\lambda_1 = -\frac{1}{\mathbf{B}_0^2}[(\mathbf{B}_0 + \mathbf{B}_{1\perp})\cdot\nabla v + \mathbf{v}_1\cdot\nabla(p_0 + \tfrac{1}{2}\mathbf{B}_0^2)] - \tfrac{1}{3}\nabla_\perp\cdot\mathbf{v}_1.$$

The equation for A is given by Eq. (4a) and the equation for $b_\parallel$, which may be derived from the parallel component of Eq. (1b), becomes

$$\frac{d}{dt}b_\parallel - (\mathbf{B}_0 + \mathbf{B}_{1\perp})\cdot\nabla v - \mathbf{v}_{1\perp}\cdot\nabla(p_0 + \mathbf{B}_0^2) - \mathbf{B}_0^2(\nabla_\perp\cdot\mathbf{v}_1) - \eta\Delta_\perp b_\parallel = 0. \tag{A3}$$

Together with the equations for ρ_1 and p_1, which may be derived from Eq. (1c) and Eq. (1d),[17] respectively, Eqs. (A1)–(A3) and (4a) form the reduced equations of the resistive g-mode fluctuations. In addition to the scaled variables and parameters introduced in Sec. III, we use

$$\tilde\lambda = -\frac{2r_0\sqrt{\rho_0}}{B_\theta|\sigma|^{5/2}}\lambda_1,\quad \tilde\rho = \frac{2}{\rho_0|\sigma|^{3/2}}\rho_1,\quad \tilde b = \frac{2}{B^2|\sigma|^{3/2}}b_\parallel,$$

$$\tilde\alpha = \frac{2r_0\sqrt{\rho_0}}{B_\theta|\sigma|^{5/2}}(\nabla_\perp\cdot\mathbf{v}_1),\quad N = \frac{-r_0\rho_0'}{\rho_0\sigma^2},$$

$$H = \frac{-2r_0(p_0 + B^2)'}{B^2\sigma^2},\quad M_\perp = \frac{\mu_\perp}{r_0B_\theta\sqrt{\rho_0}},$$

$$M_\parallel = \frac{3\mu_\parallel B_\theta|\sigma|}{r_0B^2\sqrt{\rho_0}},\quad J = \frac{r_0\sigma}{2B_\theta|\sigma|^{1/2}}(\mathbf{J}_0\cdot\mathbf{B}_0).$$

Here we note that $H = S - D$ and $D = (\gamma - 1)\Theta/\gamma + \beta N/2$. Rewriting the mode equations in terms of these scaled variables, we obtain the following set of equations of the resistive g-mode fluctuations with the parallel viscosity:

$$\frac{d\tilde A}{d\tau} = \frac{\partial\tilde\phi}{\partial\tilde\theta} + \frac{1}{R}\tilde\nabla_\perp\tilde A,$$

$$\frac{d}{d\tau}\tilde\nabla_\perp\tilde\phi = \frac{\partial}{\partial\tilde\theta}\tilde\nabla_\perp\tilde A + \{\tilde A,\tilde\Delta_\perp\tilde A\} - \frac{\partial\tilde b}{\partial y} + JM_\parallel\left(\frac{\partial\tilde\lambda}{\partial\tilde\theta} + \{\tilde A,\tilde\lambda\}\right) + M_\perp\tilde\Delta^2{}_\perp\tilde\phi,$$

$$\frac{d}{d\tau}\tilde\rho = -N\frac{\partial\tilde\phi}{\partial y} - \tilde\alpha,$$

$$\frac{d}{d\tau}\tilde b = H\frac{\partial\tilde\phi}{\partial y} + \frac{\partial\tilde v}{\partial\tilde\theta} + \{\tilde A,\tilde v\} + \tilde\alpha + \frac{1}{R}\tilde\Delta_\perp\tilde b,$$

$$\frac{d}{d\tau}\tilde p = -D\frac{\partial\tilde\phi}{\partial y} - \frac{\gamma\beta\tilde\alpha}{2} + (\gamma - 1)(K_\parallel\tilde\Delta_\parallel + K_\perp\tilde\Delta_\perp)\tilde T,$$

$$\frac{d\tilde v}{d\tau} = \frac{\partial\tilde p}{\partial\tilde\theta} + \{\tilde A,\tilde b\} + D\frac{\partial\tilde A}{\partial y} + M_\parallel\left(\frac{2}{3}\frac{\partial\tilde\lambda}{\partial\tilde\theta} + \frac{1}{3}\{\tilde A,\tilde\lambda\}\right) + M_\perp\tilde\Delta_\perp\tilde v,$$

where

$$\tilde b = \tilde p - \tilde M_\parallel\tilde\lambda,$$

$$(\beta/2)\tilde\rho = \tilde p - [(\gamma - 1)/\gamma]\tilde T,$$

$$\tilde\lambda = \frac{\partial\tilde v}{\partial\tilde\theta} + \{\tilde A,\tilde v\} + \frac{S}{2}\frac{\partial\tilde\phi}{\partial y} + \frac{1}{3}\tilde\alpha.$$

These equations are the extension of Eqs. (8) in Sec. III.

APPENDIX B: NONLINEAR STABILITY ANALYSIS OF A TOKAMAK PLASMA

In this Appendix, we will discuss some fundamental properties, particularly nonlinear stability, of the solutions of Eqs. (10) mathematically in order to see some similarity between the resistive g modes and the Bénard convection in fluid dynamics. We note that, unlike the analysis in Secs. V and VI, we will consider three-dimensional problems, i.e., $\tilde A$, $\tilde\phi$, and $\tilde p$ in Eqs. (10) depending on $(\tilde x,\tilde y,\tilde z)$ as well as τ. Therefore, the conclusions drawn here apply not only to the steady convection solutions on a single rational surface, but also to the fully developed turbulence of mulitple helicity modes. The stability of the system (10) is defined[8] as the stability of the null solutions $\tilde A = \tilde\phi = \tilde p \equiv 0$, which represents the equilibrium state of the plasma, in the following way: Let $\mathscr{E}(\tau)$ be the energy norm of the system defined as

$$\mathscr{E}(\tau) = \tfrac{1}{2}\langle|\widetilde{\nabla}_\perp \widetilde{A}|^2 + |\widetilde{\nabla}_\perp \tilde{\phi}|^2 + |\tilde{p}|^2\rangle.$$

The null solution of the system (10) is called asymptotically stable (in the sense of Lyapunov) if there is a positive number δ_I such that, for any initial condition the energy norm of that is less than δ_I, i.e., for $\mathscr{E}(0) < \delta_I$, the energy norm approaches zero for large τ, i.e., $\mathscr{E}(\tau) \to 0$ as $\tau \to \infty$. In particular, if δ_I is finite, it is called conditionally stable, and if $\delta_I = \infty$, then it is called globally stable. The globally stable null solution is also called monotonically stable if $d\mathscr{E}(\tau)/d\tau < 0$ for all $\tau > 0$.

The stabilities defined above are based on the nonlinearity of the system. For a perturbation with infinitesimal amplitude, however, we have the spectral problem of the linearized system of Eqs. (10). Assuming that the solutions of the linearized system depend exponentially on τ as $e^{q\tau}$, where q is a complex number, we replace $\partial/\partial\tau$ by q. Provided that there exist such numbers q for which the linearized system has nontrivial solutions, the numbers q are called the eigenvalues of the system. The linear stability of the null solution is then defined in connection with the eigenvalues q as follows: The null solution is called linearly stable if there are no eigenvalues such that $\mathrm{Re}\, q > 0$; marginally stable if there is at least one eigenvalue with $\mathrm{Re}\, q = 0$ and all the other eigenvalues have $\mathrm{Re}\, q \leqslant 0$; and unstable if at least one eigenvalue has $\mathrm{Re}\, q > 0$.

Before examining the stability properties, we specify the boundary conditions of Eqs. (10) more precisely. In the coordinate system $(\tilde{x},\tilde{y},\tilde{z})$, since $\tilde{x} = 0$ is taken to be a rational surface, the particular mean field line we consider on this surface comes back to its original position after a finite length $\tilde{z} = L_z$. Thus we impose the following boundary conditions:

(I) $\tilde{\phi}, \widetilde{\Delta}_\perp \tilde{\phi}, \widetilde{A}, \tilde{p}, \tilde{v}, \widetilde{T} = 0$ at $|\tilde{x}| = \delta_x$,

(II) $\tilde{\phi}, \widetilde{\Delta}_\perp \tilde{\phi}, \widetilde{A}, \tilde{p}, \tilde{v}$, and $\widetilde{T}$ and their $\tilde{y}$ derivatives are periodic with period $2\delta_y$ in $\tilde{y}$,

and

(III) $\tilde{\phi}, \widetilde{\Delta}_\perp \tilde{\phi}, \widetilde{A}, \tilde{p}, \tilde{v}$, and $\widetilde{T}$ are periodic with period L_z in $\tilde{z}$.

Here, δ_x, δ_y, and L_z are the prescribed positive values such that $\delta \ll \delta_x \ll a$, $\delta_y \simeq \delta$, and $L_z/a \geqslant O(1)$, where δ is the small scale parameter introduced in Sec. III and a is the plasma minor radius. In the linear theory in Ref. 7, δ_x/δ is taken to be ∞, in which case the asymptotic behavior of the linear solution at large $|\tilde{x}|$ (i.e., $|\tilde{x}| \gg \delta$) takes an exponentially decaying form. We presume from this linear asymptotic analysis that the solutions of the nonlinear system (10) are essentially independent of the choice of δ_x and decay rapidly as $|\tilde{x}|$ becomes large, as long as δ_x/δ is taken to be large enough. (As pointed out in Sec. VI, this is found to be the case.) The finiteness of δ_x, however, makes the mathematical treatment of the system easier. For a mathematical discussion of these boundary conditions, the readers are referred to Appendix B of Ref. 17.

In Sec. V, we discuss the case where the null solution of the system (10) is linearly unstable. The following proposition asserts that $D \leqslant 0$ is a sufficient condition for linear stability. In other words, the system may be linearly unstable only when $D > 0$.

Proposition 1: Suppose R, M, and χ are all positive constants. If $D \leqslant 0$, then $\mathrm{Re}\, q < 0$.

Proof: The linearized equations of the system (10) are

$$q\widetilde{A} = \frac{\partial\tilde{\phi}}{\partial\tilde{\theta}} + R^{-1}\widetilde{\Delta}_\perp \widetilde{A}, \tag{B1a}$$

$$q\widetilde{\Delta}_\perp \tilde{\phi} = \frac{\partial\widetilde{\Delta}_\perp \widetilde{A}}{\partial\tilde{\theta}} - \frac{\partial\tilde{p}}{\partial y} + M\widetilde{\Delta}_\perp^2 \tilde{\phi}, \tag{B1b}$$

$$q\tilde{p} = -D\frac{\partial\tilde{\phi}}{\partial y} + \chi\widetilde{\Delta}_\perp \tilde{p}. \tag{B1c}$$

Here, we allow $\widetilde{A}, \tilde{\phi}, \tilde{p}$, and q to take complex values. Adding up $\widetilde{\Delta}_\perp A \times$ Eq. (B1a)* and $\tilde{\phi}^* \times$ Eq. (B1b), and taking the $\langle\ \rangle$ average of the resulting equation, we obtain

$$q^*\langle|\widetilde{\nabla}_\perp \widetilde{A}|^2\rangle + q\langle|\widetilde{\nabla}_\perp \tilde{\phi}|^2\rangle = \left\langle \tilde{p}\frac{\partial\tilde{\phi}^*}{\partial y}\right\rangle - R^{-1}\langle|\widetilde{\Delta}_\perp \widetilde{A}|^2\rangle - M\langle|\widetilde{\Delta}_\perp \tilde{\phi}|^2\rangle. \tag{B2}$$

Here, * denotes the complex conjugate. Similarly, by multiplying Eq. (B1c)* by $\tilde{p}$ and taking the $\langle\ \rangle$ average, we have

$$q^*\langle|\tilde{p}|^2\rangle = -D\,\tilde{p}\left\langle\tilde{p}\frac{\partial\tilde{\phi}^*}{\partial y}\right\rangle - \chi\langle|\widetilde{\nabla}_\perp \tilde{p}|^2\rangle. \tag{B3}$$

If $D = 0$, then the real part of Eq. (B3) becomes

$$(\mathrm{Re}\, q)\langle|\tilde{p}|^2\rangle = -\chi\langle|\widetilde{\nabla}_\perp \tilde{p}|^2\rangle.$$

If $\langle|\widetilde{\nabla}_\perp \tilde{p}|^2\rangle \neq 0$, then $\mathrm{Re}\, q < 0$. If $\langle|\widetilde{\nabla}_\perp \tilde{p}|^2\rangle = 0$, then $\tilde{p}$ is equal to 0 almost everywhere because of the boundary conditions and the real part of Eq. (B2) becomes

$$(\mathrm{Re}\, q)(\langle|\widetilde{\nabla}_\perp \widetilde{A}|^2\rangle + \langle|\widetilde{\nabla}_\perp \tilde{\phi}|^2\rangle) = -R^{-1}\langle|\widetilde{\Delta}_\perp \widetilde{A}|^2\rangle - M\langle|\widetilde{\Delta}_\perp \tilde{\phi}|^2\rangle.$$

Therefore, $\mathrm{Re}\, q$ is negative for any nontrivial solutions. If $D < 0$, then eliminating the term $\langle\tilde{p}\partial\tilde{\phi}^*/\partial y\rangle$ from Eqs. (B2) and (B3) and taking the real part of the resulting equation yields

$$-(\mathrm{Re}\, q)[\langle|\widetilde{\nabla}_\perp \widetilde{A}|^2\rangle + \langle|\widetilde{\nabla}_\perp \tilde{\phi}|^2\rangle - (1/D)\langle|\tilde{p}|^2\rangle] = R^{-1}\langle|\widetilde{\Delta}_\perp \widetilde{A}|^2\rangle + M\langle|\widetilde{\Delta}_\perp \tilde{\phi}|^2\rangle - (\chi/D)\langle|\widetilde{\nabla}_\perp \tilde{p}|^2\rangle. \tag{B4}$$

Therefore, for the nontrivial solutions of the system (B1), we have $\mathrm{Re}\, q < 0$.

(Q.E.D.)

We now turn to the nonlinear modes. In Sec. IV, we discuss some properties of the solutions that satisfy the condition that the time averages of $d\langle|\widetilde{\nabla}_\perp \widetilde{A}|^2\rangle/d\tau$, $d\langle|\widetilde{\nabla}_\perp \tilde{\phi}|^2\rangle/d\tau$, and $d\langle|\tilde{p}|^2\rangle/d\tau$ vanish. The saturated modes and stationary turbulence are "nontrivial" examples of such solutions: $\langle|\widetilde{\nabla}_\perp \widetilde{A}|^2\rangle$, $\langle|\widetilde{\nabla}_\perp \tilde{\phi}|^2\rangle$, and $\langle|\tilde{p}|^2\rangle$ are bounded in time (so that $\overline{d\mathscr{E}(\tau)/d\tau} = 0$), but none of them vanish at $\tau \to +\infty$. However, this kind of solution may not exist for a certain range of parameters of R, M, χ, and D. In fact, the following proposition asserts that if the heat conductivity

χ is taken to be 0, then there is no such saturated mode nor stationary turbulence; the solutions of the system (10) with $\chi = 0$ either keep growing or decay to a trivial state.

Proposition 2: Suppose R and M are positive constants, $\chi = 0$, and $D \neq 0$. If $\mathscr{E}(\tau)$ is bounded in time τ, then $\langle|\widetilde{\nabla}_\perp \widetilde{A}|^2\rangle$, $\langle|\widetilde{\nabla}_\perp \tilde{\phi}|^2\rangle$, and $\langle\tilde{p}\partial\tilde{\phi}/\partial y\rangle \to 0$ as $\tau \to +\infty$. In other words, the condition that $\chi > 0$ is essential in order for Eqs. (10) to have nontrivial saturated modes or stationary turbulence solutions.

Proof: Before proving this proposition, we recall that $\langle\tilde{p}\partial\tilde{\phi}/\partial y\rangle$ is proportional to the anomalous heat flux. Therefore, this proposition asserts that if $\chi = 0$ and $\mathscr{E}(\tau)$ is bounded, then the time average of the anomalous heat flux is zero and the functions $\widetilde{A}$ and $\tilde{\phi}$ decay to zero almost everywhere. We also point out that it is possible to construct a solution $\tilde{p}$ approaching a nonzero (almost arbitrary) function as $\tau \to \infty$, which indicates that the assumption that $\chi = 0$ violates the uniqueness[17] of the solutions of this system. Therefore, the condition that $\chi > 0$ is essential in order for Eqs. (10) to be a valid model of a physical system.

From now on, we again consider real-valued solutions $\widetilde{A}$, $\tilde{\phi}$, and $\tilde{p}$ of the system (10). From the assumption, the time averages of $d\langle|\widetilde{\nabla}_\perp \widetilde{A}|^2\rangle/d\tau$, $d\langle|\widetilde{\nabla}_\perp \tilde{\phi}|^2\rangle/d\tau$, and $d\langle|\tilde{p}|/d\tau$ vanish, and, therefore, we have Eqs. (12) and (15). If $\chi = 0$, then we obtain $\overline{\langle p\partial\tilde{\phi}/\partial y\rangle} = \overline{\langle|\widetilde{\Delta}_\perp \widetilde{A}|^2\rangle} = \overline{\langle|\widetilde{\Delta}_\perp \tilde{\phi}|^2\rangle} = 0$. Since $\langle|\widetilde{\Delta}_\perp \widetilde{A}|^2\rangle$ and $\langle|\widetilde{\Delta}_\perp \tilde{\phi}|^2\rangle$ are positive functions of τ, therefore, $\langle|\widetilde{\Delta}_\perp \widetilde{A}|^2\rangle$, $\langle|\widetilde{\Delta}_\perp \tilde{\phi}|^2\rangle \to 0$ as $\tau \to +\infty$.

In order to prove that $\langle|\widetilde{\nabla}_\perp \widetilde{A}|^2\rangle$ and $\langle|\widetilde{\nabla}_\perp \tilde{\phi}|^2\rangle \to 0$, we will use the Poincare inequality[20]: For any real-valued function f (with a proper differentiability condition) satisfying the boundary conditions (I), (II), and (III), there exists a positive constant C such that

$$\langle|\widetilde{\nabla}_\perp f|^2\rangle \leqslant C\langle|\widetilde{\Delta}_\perp f|^2\rangle.$$

Here we note that the constant C is finite since the domain of f is finite. Applying the Poincare inequality to $\widetilde{A}$ and $\tilde{\phi}$ and using the fact that $\langle|\widetilde{\Delta}_\perp \widetilde{A}|^2\rangle$ and $\langle|\widetilde{\Delta}_\perp \tilde{\phi}|^2\rangle \to 0$ as $\tau \to \infty$, we obtain $\langle|\widetilde{\nabla}_\perp \widetilde{A}|^2\rangle$ and $\langle|\widetilde{\nabla}_\perp \tilde{\phi}|^2\rangle \to 0$ as $\tau \to +\infty$. To prove that $\langle\tilde{p}\partial\tilde{\phi}/\partial y\rangle \to 0$ as $\tau \to +\infty$, we use Schwartz's inequality

$$\left|\left\langle\tilde{p}\,\frac{\partial\tilde{\phi}}{\partial y}\right\rangle\right|^2 \leqslant \langle|\tilde{p}|^2\rangle\left\langle\left|\frac{\partial\tilde{\phi}}{\partial y}\right|^2\right\rangle \leqslant \langle|\tilde{p}|^2\rangle\langle|\widetilde{\nabla}_\perp \tilde{\phi}|^2\rangle. \tag{B5}$$

Since $\langle|\tilde{p}|^2\rangle$ is assumed to be bounded in time τ, $\langle\tilde{p}\partial\tilde{\phi}/\partial y\rangle \to 0$ as $\tau \to +\infty$.

(Q.E.D.)

From Propositions 1 and 2, it is found that positive values of R, M, χ, and D are necessary to obtain saturated modes or stationary turbulence solutions. In the following theorem, we will show that, if the plasma is sufficiently viscous or heat conductive, the null solution is still stable even for some positive D.

Theorem 1: Suppose R, M, and χ are all positive constants. If either M or χ is large enough, there is a positive critical value D_c of D such that for any $D < D_c$, the null solution is monotonically stable.

Proof: Let H be the set of all real valued functions (with sufficient smoothness[17]) of $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$ satisfying the boundary conditions (I), (II), and (III). The solutions $\widetilde{A}$, $\tilde{\phi}$, and $\tilde{p}$ of the system (10) are obviously some elements of H. For any element f in H, we have the following Poincare inequalities as before: There exists a constant C such that

$$\langle f^2\rangle \leqslant C\langle|\widetilde{\nabla}_\perp f|^2\rangle \quad\text{and}\quad \langle|\widetilde{\nabla}_\perp f|^2\rangle \leqslant C\langle|\widetilde{\Delta}_\perp f|^2\rangle. \tag{B6}$$

By adding Eqs. (11) and (16), we obtain

$$\frac{d}{d\tau}\mathscr{E}(\tau) = -(1+D)\left\langle\tilde{p}\,\frac{\partial\tilde{\phi}}{\partial y}\right\rangle - \mathscr{F}(\tau),$$

where

$$\mathscr{F}(\tau) = \frac{1}{R}\langle|\widetilde{\Delta}_\perp \widetilde{A}|^2\rangle + M\langle|\widetilde{\Delta}_\perp \tilde{\phi}|^2\rangle + \chi\langle|\widetilde{\Delta}_\perp \tilde{p}|^2\rangle.$$

Therefore,

$$\frac{d}{d\tau}\mathscr{E}(\tau) = -\mathscr{F}(\tau)\left(1 + (1+D)\,\frac{\langle\tilde{p}\,\partial\tilde{\phi}/\partial y\rangle}{\mathscr{F}(\tau)}\right).$$

Let us define D_c as

$$\sup_{\widetilde{A},\tilde{\phi},\tilde{p}\in H}\left(-\frac{\langle\tilde{p}\,\partial\tilde{\phi}/\partial y\rangle}{\mathscr{F}(\tau)}\right) \equiv \frac{1}{1+D_c}. \tag{B7}$$

Since $|\langle\tilde{p}\,\partial\tilde{\phi}/\partial y\rangle/\mathscr{F}|$ is bounded from above, we have $D_c > -1$. In fact, this boundedness is shown as follows. We have

$$\left|\frac{\langle\tilde{p}\,\partial\tilde{\phi}/\partial y\rangle}{\mathscr{F}(\tau)}\right|^2 \leqslant \frac{\langle\tilde{p}\,\partial\tilde{\phi}/\partial y\rangle^2}{(M\langle|\widetilde{\Delta}_\perp \tilde{\phi}|^2\rangle + \chi\langle|\widetilde{\nabla}_\perp \tilde{p}|^2\rangle)^2}. \tag{B8}$$

From Schwartz's inequality (B5) and Poincare's inequalities (B6) for $\tilde{p}$ and $\widetilde{\nabla}_\perp \tilde{\phi}$, the right-hand side of inequality (B8) is less than

$$C^2\,\frac{\langle|\widetilde{\nabla}_\perp \tilde{p}|^2\rangle\langle|\widetilde{\Delta}_\perp \tilde{\phi}|^2\rangle}{(M\langle|\widetilde{\Delta}_\perp \tilde{\phi}|^2\rangle + \chi\langle|\widetilde{\nabla}_\perp \tilde{p}|^2\rangle)^2} \leqslant \frac{C^2}{2\chi M}.$$

It follows that the supreme value given by Eq. (B7) can be less than 1 by taking either χ or M to be large enough. Therefore, with such a choice of χ and M, we obtain $D_c > 0$.

Continuing our proof, we obtain from Eq. (B7)

$$\frac{d}{d\tau}\mathscr{E}(\tau) \leqslant -\mathscr{F}(\tau)\left(1 - \frac{1+D}{1+D_c}\right). \tag{B9}$$

Let $\Lambda = \min(1/R, M, \chi)$, then, by using Poincare's inequalities (B6) for $\widetilde{\nabla}_\perp \widetilde{A}$, $\widetilde{\nabla}_\perp \tilde{\phi}$, and $\tilde{p}$, we have

$$\begin{aligned}\mathscr{F}(\tau) &= \Lambda\left(\frac{1}{R\Lambda}\langle|\widetilde{\Delta}_\perp \widetilde{A}|^2\rangle + \frac{M}{\Lambda}\langle|\widetilde{\Delta}_\perp \tilde{\phi}|^2\rangle + \frac{\chi}{\Lambda}\langle|\widetilde{\nabla}_\perp \tilde{p}|^2\rangle\right)\\ &\geqslant \Lambda(\langle|\widetilde{\Delta}_\perp \widetilde{A}|^2\rangle + \langle|\widetilde{\Delta}_\perp \tilde{\phi}|^2\rangle + \langle|\widetilde{\nabla}_\perp \tilde{p}|^2\rangle)\\ &\geqslant 2\Lambda C^{-1}\cdot\mathscr{E}(\tau).\end{aligned} \tag{B10}$$

From inequalities (B9) and (B10), we obtain

$$\frac{d}{d\tau}\mathscr{E}(\tau) \leqslant -\frac{2\Lambda}{C}\mathscr{E}(\tau)\left(1 - \frac{1+D}{1+D_c}\right),$$

or, by integrating, we obtain

$\mathscr{E}(\tau) \leqslant \mathscr{E}(0)\exp\{-(2\Lambda/C)[1-(1+D)/(1+D_c)]\tau\}$.

Hence, if $D < D_c$, then the perturbation decays exponentially with a decay constant proportional to the smallest dissipation constant $\Lambda = \min(1/R, M, \chi)$. (Q.E.D.)

Since $|\langle\tilde{p}\,\partial\tilde{\phi}/\partial y\rangle/\mathscr{F}|$ is bounded from above, there exists the least upper bound of $|\langle\tilde{p}\,\partial\tilde{\phi}/\partial y\rangle/\mathscr{F}|$. Suppose $\widetilde{A}_0$, $\tilde{\phi}_0$, and $\tilde{p}_0$ take the maximum value of $-\langle\tilde{p}\partial\tilde{\phi}/\partial y\rangle/\mathscr{F}$. (We note that $\widetilde{A}_0$, $\tilde{\phi}_0$, and $\tilde{p}_0$ may not have enough smoothness but they can be approximated by functions in H as "closely" as possible.[17]) By choosing $\widetilde{A}_0$, $\tilde{\phi}_0$, and $\tilde{p}_0$ as an initial condition, we have, from Eq. (B7),

$$\frac{d}{d\tau}\mathscr{E}(0) = -\mathscr{F}(0)\left(1 - \frac{1+D}{1+D_c}\right).$$

It follows that $d\mathscr{E}(0)/d\tau \geqslant 0$ if $D \geqslant D_c$. Therefore, the following corollary holds.

Corollary: Under the condition of Theorem 1, define D_c by Eq. (B7). Then, $D < D_c$ is the necessary and sufficient condition for monotonic stability.

We call this critical value D_c the energy stability limit. We also define the linear stability limit D_L as the value that makes the null solution of the system linearly marginally stable. The following theorem gives the relationship between the energy stability limit D_c and the linear stability limit D_L.

Theorem 2: Suppose R, M, and χ are all positive constants. Then $D_c \leqslant D_L$.

Proof: Adding up Eqs. (B2) and (B3) and taking the real part of the resulting equation, we obtain

$$0 = (1 + D_L)\,\mathrm{Re}\left\langle \tilde{p}\frac{\partial\tilde{\phi}^*}{\partial y}\right\rangle + R^{-1}\langle|\widetilde{\Delta}_\perp\widetilde{A}|^2\rangle + M\langle|\widetilde{\Delta}_\perp\tilde{\phi}|^2\rangle + \chi\langle|\tilde{p}|^2\rangle,$$

where we used the condition that $\mathrm{Re}\,q = 0$ if $D = D_L$. Therefore, we have

$$\frac{1}{1+D_L} = \frac{-\langle \tilde{p}_r\,\partial\tilde{\phi}_r/\partial y + \tilde{p}_i\,\partial\tilde{\phi}_i/\partial y\rangle}{R^{-1}\langle|\widetilde{\Delta}_\perp\widetilde{A}|^2\rangle + M\langle|\widetilde{\Delta}_\perp\tilde{\phi}|^2\rangle + \chi\langle|\widetilde{\nabla}_\perp\tilde{p}|^2\rangle}. \tag{B11}$$

Here, the subscripts r and i denote the real and imaginary parts of the quantity, respectively. Writing $\langle|\widetilde{\Delta}_\perp\tilde{\phi}|^2\rangle = \langle|\widetilde{\Delta}_\perp\tilde{\phi}_r|\rangle + \langle|\widetilde{\Delta}_\perp\tilde{\phi}_i|^2\rangle$ and $\langle|\widetilde{\nabla}_\perp\tilde{p}|^2\rangle = \langle|\widetilde{\nabla}_\perp\tilde{p}_r|^2\rangle + \langle|\widetilde{\nabla}_\perp\tilde{p}_i|^2\rangle$ and applying to Eq. (B11) the inequality

$$\frac{\Sigma_{i=1}^2 a_i}{\Sigma_{j=1}^2 b_j} \leqslant \max_{1\leqslant i,j\leqslant 2}\left(\frac{a_i}{b_j}\right),$$

where b_j are positive, we obtain

$$\frac{1}{1+D_L} \leqslant \sup_{\tilde{\phi},\tilde{p}\in H} \frac{-\langle \tilde{p}\,\partial\tilde{\phi}/\partial y\rangle}{M\langle|\widetilde{\Delta}_\perp\tilde{\phi}|^2\rangle + \chi\langle|\widetilde{\nabla}_\perp\tilde{p}|^2\rangle}$$
$$= \sup_{\widetilde{A},\tilde{\phi},\tilde{p}\in H} -\frac{\langle \tilde{p}\,\partial\tilde{\phi}/\partial y\rangle}{\mathscr{F}(\tau)}$$
$$= 1/(1+D_c).$$

It follows that $D_c \leqslant D_L$. (Q.E.D.)

To conclude this appendix, it is found that we need positive parameters of D as well as R, χ, M in the system (10) in order to find the solutions that grow from small initial values and eventually saturate with finite amplitude or lead to stationary turbulence. It is also found that, with a proper choice of the viscosity M and the heat conductivity χ, there exists a positive energy stability limit D_c, which is smaller than or equal to the linear stability limit D_L. This shows that, if the plasma is sufficiently viscous or heat conductive, the null solution is stable (to any perturbation) even for some positive D. Since D is the only free energy source of the system, if $D < D_c$, then all the energy fed to the modes by the mean pressure gradient dissipates through the collisional diffusion, and not through convection. When $D > D_L$, then even an infinitesimal perturbation given to the null solution starts to grow and the free energy produced from the mean pressure gradient is transferred by convective motion as well as the collisional diffusion. We thus see some similarity between the resistive g modes of the system (10) and the Bénard convection in fluid dynamics. Here we note that the parameter D corresponds to the Rayleigh number of the Bénard convection.

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